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SPINNING SYMMETRIC MISSILES

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by

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ABSTRACT

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The case of a spinning symmetric body flying at a constant speed where the aerodynamic restoring moment is an arbitrary function of the angle of attack is examined. The analysis is two-fold: first, the general problem is discussed, in suitable non-dimensional form, to establish the generalized stability boundaries and, second, the attention is directed to the inverse problem whereby the pertinent aerodynamic parameters are extracted from a given bounded solution, suitable for use in data reduction. The general case of non-planar motion is examined and shown to be analogous to the classical orbital problems, differing only in the form of the governing potential function. The general solution is obtained in integral form and the special cases of linear aerodynamics and cubic restoring moments have been integrated and studied to reveal all the pertinent characteristics. The various combinations of potential, initial conditions and angular momentum (including that due to the impressed spin rate) are shown to determine whether or not the motion is planar, circular, elliptic or non-conic, stable or unstable and the various cases are categorized to aid in the prediction of the motion of spinning symmetrical bodies acting under non-linear aerodynamic restoring moments.

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I. INTRODUCTION

Up to fairly recent years the underlying design criteria for the dynamic free-flight motion of ballistic missiles and rockets, both guided and unguided, have had as their basis, the linear theory. Such a theory restricts all forces and moments acting on the aerodynamic body to be proportional to the deviations from the flight path. In many cases, the linear theory may be applied with impunity and the subsequent rocket flight is a success. However, embarrassing failures on some flights prompted a deeper analysis of the spinning aerodynamic body problem (Refs. (1) - (5)) in an attempt to isolate the destabilizing influences and to enable the designs to circumvent such failures.

The present paper attacks the general problem of the spinning bodies in a non-linear aerodynamic field and by suitable transformation techniques shows that it is related to the classical problems in orbital mechanics. Such an analogy allows for a generalized treatment which allows for the governing parameters to be revealed succinctly. The problem is considered first from a topological concept to establish the stability boundaries and secondly the analysis has been quantized so as to provide the data analyst means by which a given bounded solution, or trajectory, may be analyzed to extract the pertinent aerodynamic parameters, both linear and non-linear.

Certain restrictions have been placed upon the analysis to allow for tractable solutions: the motion is assumed to be conservative and the non-linear restoring moment is, for the most part, assumed to be a two-term polynomial in which the power and magnitude of the non-linear term is left arbitrary. It must be further noted that in the body of the report certain real phenomena are ignored and prudence must be exercised in the application of the analysis in any particular case. It is assumed that throughout the flight of the (constant) spinning missile the velocity remains constant together with the density of the air and further that the oncoming stream is uniform with no gusts causing random disturbances to be applied to the body. Such disturbances could destroy any phase coherency of an otherwise periodic motion.

II. THE GENERAL PROBLEM

The free-flight motion of a spinning aerodynamic symmetric body is but a special case of the classical rigid body vibration problem frequently encountered in physics. The idiosyncracy of the aerodynamic problem lies in the form of the potential from which the inherent forces are derived. In this Section will be given the basic assumptions and analysis leading to the generalized and normalized equations of motion and their integral solutions from which the succeeding Sections may extract the pertinent results for further analysis.

A. Basic Equations of Motion

For the purposes of continuity and reference, the equations of motion will be derived from Newton's Laws. It will be assumed that the body has six degrees of freedom defined vectorially as,

$$\vec{u}(t) = (U + u,\ v,\ w) \quad ; \quad \vec{\omega}(t) = (p,\ q,\ r) \tag{2.1}$$

The vectors $\vec{u}(t)$, $\vec{\omega}(t)$ pertain to a body-fixed axis system. Fig. 1 serves to describe the system and symbols used in the text. The angular momentum $\vec{L} = \Phi\vec{\omega}$ where, by virtue of the choice of a principal body-axis system, the inertia tensor Φ is diagonal and given by,

$$\Phi = \begin{pmatrix} I_3 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \tag{2.2}$$

The aerodynamic forces and moments are,

$$\vec{F}(t) = (X, Y, Z) \quad ; \quad \vec{N}(t) = (L, M, N) \tag{2.3}$$

For this study it will be assumed that the aerodynamic forces are linear functions of the perturbations. The aerodynamic moments will be assumed expressible in uncoupled static and dynamic components of which only the static restoring moments will be assumed to be non-linear functions of the perturbations. The exact form of these functional relationships is to be given.

Newton's equations, expressed in the body-fixed system are,

$$\left(\frac{d}{dt} + \vec{\omega}\times\right)\left\{u\right\} = \vec{F}(t) + m\vec{g} \qquad (2.4)^*$$

$$\left(\frac{d}{dt} + \vec{\omega}\times\right)\left\{L\right\} = \vec{N}(t) \tag{2.5}$$

where in the operator on the left hand side of the equations the time derivative is with respect to the body-fixed system. The basic equations then assume the form

$$m\begin{Bmatrix} \dot{u} + qw - rv \\ \dot{v} + r(U+u) - pw \\ \dot{w} + pv - q(U+u) \end{Bmatrix} = \begin{Bmatrix} X(\alpha, \beta, \ldots) \\ Y(\alpha, \beta, \ldots) \\ Z(\alpha, \beta, \ldots) \end{Bmatrix} + m\begin{Bmatrix} g_1 \\ g_2 \\ g_3 \end{Bmatrix} \tag{2.6}$$

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* where $\vec{g}$ represents the gravity vector.

and

$$I \begin{Bmatrix} \mu P \\ \dot{q} - (1-\mu)pr \\ \dot{r} + (1-\mu)pq \end{Bmatrix} = \begin{Bmatrix} L(\alpha, \beta, \ldots) \\ M(\alpha, \beta, \ldots) \\ N(\alpha, \beta, \ldots) \end{Bmatrix} \qquad (2.7)$$

where $\mu = \dfrac{I_3}{I}$, the slenderness parameter and $\alpha = \tan^{-1}(w/U)$ and $\beta = \tan^{-1}(v/U)$.

Before equations (2.6) and (2.7) can be simplified further the specific nature of $\vec{F}(t)$ and $\vec{N}(t)$ must be given. From the assumption of symmetry and linearity the forces are expressed as,

$$Y(\alpha, \beta, \ldots) = Z_\alpha \beta \qquad (2.8)$$

$$Z(\alpha, \beta, \ldots) = Z_\alpha \alpha \qquad (2.9)$$

It will be further assumed that the forward speed remains constant throughout that portion of flight under consideration (U = constant) which eliminates the need for the $X(\alpha, \beta, \ldots)$ equation and reduces the problem to one in five degrees of freedom. This does not preclude a discussion of the dissipative problem but does imply that the analysis will apply most directly to slender bodies of low resistance. For the main part of the study the roll rate will be assumed constant at some specified rate (p = constant), or, at best, as a slowly varying function in time; in which case the need to specify the $L(\alpha, \beta, \ldots)$ equation is also avoided. This reduces the problem, not too restrictively, to one of four degrees of freedom (α, β, q, r).

The main objective is to study the effects of a non-linear static restoring moment on the system in regard to both stability boundaries and the nature of the solutions. A further objective is to enable the pertinent parameters of the system to be revealed given the physical bounded solution.

The static moment will be separated from the dynamic moment in the form,

$$\vec{N}(\alpha, \beta, \ldots) = \vec{N}_S(\alpha, \beta, \ldots) + \vec{N}_D(\dot{\alpha}, \dot{\beta}, \ldots) \qquad (2.10)$$

where the dynamic moment is assumed to be linear in the velocity perturbations:

$$\vec{N}_D(\dot{\alpha}, \dot{\beta}, \ldots) = \begin{Bmatrix} L_D \\ M_q q + M_{\dot{\alpha}} \dot{\alpha} + \ldots \\ M_q r + M_{\dot{\alpha}} \dot{\beta} + \ldots \end{Bmatrix} \qquad (2.11)$$

If it is further assumed that the airflow over the body is symmetric such that the non-linearity in the static restoring moment may be expressed in terms of the resultant angle of attack $(\alpha^2 + \beta^2)^{\frac{1}{2}}$ *, then a polynomial representation for $\vec{N}_S(\alpha, \beta, \ldots)$ is given by,

$$\vec{N}_S(\alpha, \beta) = (-1)^j I |M_o| (\alpha \vec{i} + \beta \vec{j}) - I \sum_{n=1}^{N} (\alpha \vec{i} + \beta \vec{j})(\alpha^2 + \beta^2)^{\frac{n+m}{2}} M_n \qquad (2.12)$$

where $\vec{i}, \vec{j}$ are unit vectors along the perturbation α, β axes respectively. The potential from which the restoring moment may be

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* which implies small angles.

derived must satisfy,

$$\nabla U(\alpha, \beta) = \vec{N}(\alpha, \beta) \tag{2.13}$$

It can be shown that such a potential exists and is found to be,

$$U(\alpha, \beta) = \tfrac{1}{2}(-1)^j I|M_o| (\alpha^2+\beta^2) - \sum_{n=1}^{N} \frac{IM_n}{n+m+2} (\alpha^2+\beta^2)^{\frac{n+m+2}{2}} \tag{2.14}$$

The quantity m (= 1, 2, ...) is an arbitrary integer. From physical considerations it is seen that a characteristic of the static restoring moment representation is that it is anti-symmetric with respect to the origin (or trim condition). This property implies that $\vec{N}_S(\alpha, \beta)$ as given by (2.12) would necessarily consist of a power series in odd powers of the resultant angle of attack. Graphically, the restoring moments under consideration may be presented as shown in Fig. 2.

Note: $(-1)^j |M_o|$ is the familiar linear static stability parameter where,

$j = 0$ corresponds to initial static instability

$j = 1$ corresponds to initial static stability

For this paper, only the first term in the polynomial representation will be retained (i.e., $r = 1$) but m will remain arbitrary. The special case $m = 1$ the cubic (plus linear) restoring moment will receive particular attention. With these assumptions, the equations of motion reduce to,

$$\ddot{\beta} + H\dot{\beta} + \nu_1\beta - (2-\mu)|M_o|^{\frac{1}{2}} p\dot{\alpha} + K_1|M_o|^{\frac{1}{2}} p\alpha + M_1\beta(\alpha^2+\beta^2)^{\frac{1+m}{2}} = \dot{Q}_1(t) \tag{2.15}$$

$$\ddot{\alpha} + H\dot{\alpha} + \nu_1\alpha + (2-\mu)|M_o|^{\frac{1}{2}} p\dot{\beta} - K_1|M_o|^{\frac{1}{2}} p\beta + M_1\alpha(\alpha^2+\beta^2)^{\frac{1+m}{2}}$$

$$= \dot{Q}_2(t) \qquad (2.16)$$

These two second-order ordinary differential equations will describe the motion of the spinning body in the α, β (cross-flow) plane. In this plane the motion will resemble that shown in Fig. 3. The system is one of four degrees of freedom where the remaining two q, r have been algebraically eliminated but may be recalled from the subset of (2.15) and (2.16), viz:

$$r(t) = p\alpha - \dot{\beta} + \frac{Z_\alpha}{mU}\beta + G_2(t) \qquad (2.17)$$

$$q(t) = \dot{\alpha} + p\beta - \frac{Z_\alpha}{mU}\alpha + G_1(t) \qquad (2.18)$$

where $G_1(t) = \frac{g_3}{U} + \frac{Q_1(t)}{mU}$; $G_2 = \frac{g_2}{U} + \frac{Q_2(t)}{mU}$ (2.19)

For the particular system under study namely free-motion $\dot{Q}_1(t) = \dot{Q}_2(t) = 0$. Further it will be taken that $\dot{p} = 0$ i.e. p = constant, or at best a slow function of time and also that $\frac{M_q}{I} \cdot \frac{Z_\alpha}{mU} \ll |M_o|$, the usual aerodynamic case. The various coefficients in (2.15) and (2.16) are related to the aerodynamic physical quantities by the following,

$$H = \frac{\rho US}{2m} \left\{ C_{N_\alpha} - \left(\frac{\ell}{K}\right)^2 \left[C_{m_q} + C_{m_{\dot{\alpha}}}\right]\right\} \qquad (2.20a)$$

$$M_o = \frac{\rho US}{2m} \ell K^{-2} \left|C_{m_\alpha}(0)\right| \qquad (2.20b)$$

$$K_1 = \frac{1}{I} \left\{ M_q + M_{p\beta} \right\} + \frac{Z_a}{mU} (1 - \mu) \tag{2.20c}$$

$$\nu_1 = -(-1)^j |M_o| - (1-\mu) p^2 + \frac{M_q}{I} \frac{Z_a}{mU} \tag{2.20d}$$

M_1, m = amount and degree of non-linearity.

The dots denote differentiation with respect to real time.

B. Non-Dimensionalized Form of Equations

To avoid the occurrence of an imaginary time scale for the case $j = 0$ the equations will not be completely normalized but merely distorted by the linear frequency parameter $|M_o|^{\frac{1}{2}}$. From a suitable dimension analysis the following non-dimensional quantities may be formed,

$$\delta = \frac{H}{|M_o|^{\frac{1}{2}}} \quad ; \quad \epsilon = \frac{M_1 a_m^{1+m}}{|M_o|^{\frac{1}{2}}} \quad ; \quad P = \frac{p}{|M_o|^{\frac{1}{2}}} \tag{2.21a}$$

$$a = \frac{a}{a_m} \quad ; \quad \beta = \frac{\beta}{a_m} \quad ; \quad \tau = |M_o|^{\frac{1}{2}} t \tag{2.21b}$$

Also, for convenience, let $1 - \mu = K$. There should be no confusion on the use of $a\beta$ for the non-dimensional as well as dimensional perturbations. In normalized form the basic equations of motion become,

$$\ddot{\beta} + \delta\dot{\beta} - \nu^2\beta - (1+K)\,P\dot{\alpha} + K_1 P\alpha + \epsilon\beta(\alpha^2+\beta^2)^{\frac{1+m}{2}} = 0 \qquad (2.22a)$$

$$\ddot{\alpha} + \delta\dot{\alpha} - \nu^2\alpha + (1+K)\,P\dot{\beta} - K_1 P\beta + \epsilon\alpha(\alpha^2+\beta^2)^{\frac{1+m}{2}} = 0 \qquad (2.22b)$$

where, now, the dots denote differentiation with respect to normalized time, τ, and the linear frequency parameter ν^2 is given by

$$\nu^2 = (-1)^j + KP^2 \qquad (2.23)^*$$

Equations (2.22) form the basic set, in normalized form, from which the analysis will evolve to yield solutions and criteria for the general motion. Even in the linear theory ($\epsilon = 0$) the complete solution for $\alpha(\tau)$ and $\beta(\tau)$ involve the superposition of oscillations and is only conditionally periodic; for unless the resonant frequencies are commensurable $\alpha(\tau)$ and $\beta(\tau)$ will never repeat themselves. It will be shown that only for special conditions will the motion be periodic and moreover that periodic motion does not exist for $\alpha(\tau)$ and $\beta(\tau)$ in the general non-planar case. This is tantamount to saying that the physical coordinates $\alpha(\tau)$ and $\beta(\tau)$ are not the separation coordinates of the problem, which should not be surprising. These separation (or normal) coordinates, which by definition, are periodic may be obtained by a contact transformation and this will be done such that the problem becomes much more tractable.

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* It might be thought from Eq. (2.23) that a resonance value for P exists for the linear aerodynamically stable case but this is not the case as will be shown when the equations are expressed in the proper coordinate system.

Also, in order to study the boundedness of the solutions and other useful insight the polar form for these equations will be obtained. In such a coordinate system, which it may be reasoned, is a natural choice for this symmetric problem, the resulting equations resemble those describing the motion of planets, electrons or other orbital problems encountered in physics but with an unfamiliar potential. This analogy will allow for several interesting characteristics of the aerodynamic problem to be revealed. The normal coordinates are actually a misnomer in this context since by definition they can only uncouple the linear equations. However, by decoupling the linear part of the equations of motion a helpful form of the equations is derived. These normal coordinates may be defined by the transformation equation

$$\begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} = (a_{ij}) \begin{Bmatrix} \sigma_1 \\ \sigma_2 \end{Bmatrix} \tag{2.24}$$

The transformation will consist of a rotation in the plane. Consider then the system (2.22) with $\epsilon = 0$ and put $\dot{\alpha}(\tau) = \gamma(\tau)$ and $\dot{\beta}(\tau) = \mu(\tau)$ and consider the vector $X(\tau) = (\alpha, \beta, \gamma, \mu)$. For this transformation, the system is to be described by the first-order equation in the four-dimensional X-space, given as,

$$\frac{dX}{d\tau} = AX \tag{2.25}$$

which on substituting the components gives,

$$\begin{Bmatrix} \dot{\alpha}(\tau) \\ \dot{\beta}(\tau) \\ \dot{\gamma}(\tau) \\ \dot{\mu}(\tau) \end{Bmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \nu^2 & K_1 P & -\delta & -(1+K)P \\ -K_1 P & \nu^2 & (1+K)P & -\delta \end{pmatrix} \begin{Bmatrix} \alpha(\tau) \\ \beta(\tau) \\ \gamma(\tau) \\ \mu(\tau) \end{Bmatrix} \qquad (2.26)$$

The matrix A must be diagonalized for (2. 25) to be uncoupled, which condition is determined from $|A - \lambda\delta_{ij}| = 0$. The eigenvectors may then be determined and finally it may be taken that the required transformation is (when expressed in 2-space),

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \end{Bmatrix} \equiv \begin{Bmatrix} \sigma \\ \bar{\sigma} \end{Bmatrix} = \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} \qquad (2.27)$$

Furthermore, no real form exists for the normal coordinates that are complex conjugates as seen from (2. 27). Expressed in terms of the normal coordinates the basic equations assume the form,

$$\ddot{\sigma} + \left[\delta + i(1+K)P\right]\dot{\sigma} - \left[\nu^2 + iK_1 P\right]\sigma + \epsilon\sigma|\sigma|^{1+m} = 0 \qquad (2.28a)$$

$$\ddot{\bar{\sigma}} + \left[\delta - i(1+K)P\right]\dot{\bar{\sigma}} + \left[\nu^2 - iK_1 P\right]\bar{\sigma} + \epsilon\bar{\sigma}|\sigma|^{1+m} = 0 \qquad (2.28b)$$

It will be shown later when the initial value problem is discussed, that the set (2. 28) yields periodic solutions for the non-linear system.

C. The Central Force Field Formulation

1. Energy Relations and The Orbit Equation

By a further transformation to polar coordinates the motion of the spinning body will resemble that of the single or two-body problem in physics. The origin of the force field will be the center of gravity of the aerodynamic body and the unit mass will trace out an orbit that in actuality will be the path traced out by the nose of the missile with respect to the center of gravity. The symmetry of (2.28) allows for consideration of just one normal coordinate say $\sigma(\tau)$. Its equation may be written

$$\ddot{\sigma} + A_1\dot{\sigma} + B\sigma + \epsilon\sigma\left|\sigma\right|^{1+m} = 0 \qquad (2.29)$$

with the complex aerodynamic coefficients,

$$A_1 = \delta + i(1+K)P \qquad (2.30)$$

$$B = -\left[(-1)^j + KP^2 + iK_1P\right] \qquad (2.30b)$$

Consider the transformed coordinate system,

$$\sigma = Qe^{-\frac{1}{2}A_1\tau} \qquad (2.31)$$

such that the equation of motion becomes

$$\ddot{Q} + \mathcal{L}\dot{Q} + \epsilon Q\left|Qe^{-\frac{1}{2}A_1\tau}\right|^{1+m} = 0 \qquad (2.32)$$

The equivalent frequency parameter,

$$\mathcal{L} = B - \frac{1}{4}A_1^2 \qquad (2.35)$$

It will become informative to study the motion in a physical plane, in which case the transformation,

$$Q = \rho e^{i\varphi} \tag{2.34a}$$

with

$$\rho = re^{-\frac{1}{2}A_1\tau} = (\alpha^2 + \beta^2)^{\frac{1}{2}} e^{-\frac{1}{2}A_1\tau} \tag{2.34b}$$

will culminate in the orbital equations

$$\ddot{\rho} - \rho(\dot{\varphi})^2 + \mathrm{Re}(\mathcal{L})\rho + \epsilon\rho^{m+2} = 0 \tag{2.35a}$$

$$\frac{1}{\rho}\frac{d}{d\tau}(\rho^2\dot{\varphi}) + \mathcal{J}(\mathcal{L})\rho = 0 \tag{2.35b}$$

Note: These equations (2.35) will also result from operations on the equation for the conjugate normal coordinate $\bar{\sigma}(\tau)$.

In terms of aerodynamic parameters

$$\mathrm{Re}(\mathcal{L}) = -(-1)^j + \frac{1}{4}(1-K)^2P^2 - \frac{1}{4}\delta^2 \tag{2.36b}$$

$$\mathcal{J}(\mathcal{L}) = P\left[K_1 + \frac{1}{2}\delta(1+K)\right] \tag{2.36b}$$

The closed solution form for (2.35) requires that $\mathcal{J}(\mathcal{L}) = 0$ which can only be satisfied for the cases

$$\text{(i)}\quad P = 0 \tag{2.37a}$$

$$\text{(ii)}\quad K_1 = \delta = 0 \tag{2.37b}$$

$$\text{(iii)}\quad K_1 = -\frac{1}{2}\delta(1+K) \tag{2.37c}$$

Case (i) is the case of zero spin rate which may be applicable for certain aerodynamic missiles. Case (ii) may be considered as the non-dissipative case where the aerodynamic damping becomes negligible. Many examples of slender missiles approach this case for the purposes of stability investigations. Case (iii) would be a special case where the Magnus moment contribution was nullified by the aerodynamic damping according to the relation expressed in (2.37c).

In what follows, the analysis will require that $\vartheta(\mathcal{X}) = 0$ such that equation (2.35b) integrates directly to give the conservation law,

$$\rho^2 \dot{\varphi} = A \text{ (a constant)} \tag{2.38a}$$

or when expressed in terms of the physical coordinates,

$$r^2 \dot{\theta} + \frac{1}{2}(1+K)\, Pr^2 = A \tag{2.38b}$$

This conservation law states that the combined angular momentum (A) of the inherent rotational motion in the (α, β) plane and that due to the impressed spin rate remains constant. With this in mind, the new rotational variable may be introduced in terms of the physical variables, as,

$$\varphi = \theta + \frac{1}{2}(1+K)\,\phi \tag{2.39}$$

where

$$\frac{d\phi}{d\tau} = P = \text{constant.}$$

From (2.38) comes the distinction between planar and non-planar motion*:

(1) $A = 0$ implies that for bounded motion $\dot{\varphi} = 0$ and the motion is planar in the α, β plane and the oscillations appear as a straight line orientated at some initial value $\varphi(0)$.

(2) $A \neq 0$ implies that for bounded motion $\dot{\varphi} \neq 0$ (although monotonic) and the motion will resemble Lissajous figures in the α, β plane dependent upon the magnitude and degree of non-linearity in the static restoring moment.

Since A is a constant and positive definite for this non-dissipative case then φ must be a monotonic function in time, although θ may not be, as seen from (2.39). Substitution of the conservation law (2.38) into (2.35a) will give the equivalent single degree of freedom system equation of motion, expressed in r-variable as,

$$\ddot{r} + \left[-(-1)^j + \frac{1}{4}(1-K)^2 P^2\right] \dot{r} + \epsilon r^{m+2} - \frac{A^2}{r^3} = 0 \qquad (2.40)$$

The linear static stability parameter.

$$a = -(-1)^j + \frac{1}{4}(1-K)^2 P^2 \qquad (2.41)$$

will determine the character of the linear motion. It is seen that

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* A similar classification of motions appears in Ref. 6 although the fundamental nature of the angular momentum constant is not given there per se.

for $\epsilon = 0$ the aerodynamically stable case ($j = 1$) will remain stable for all r and τ since $a > 0$. A full discussion of the stability of the motion as governed by (2.40) may be found in Section III.

Equation (2.40) shows that the general problem is described by a second-order differential equation in the class of equations $\ddot{r} + f(r) = 0$. Autonomous systems governed by such equations have received considerable attention in the literature and the basic analysis in this paper will revolve around such an equation and its manifestations.

The equation of motion (2.40) may be integrated once to yield the conservation of energy law:

$$\frac{1}{2}(\dot{r})^2 + U_1(r) = E \tag{2.42}$$

which shows the analogy of the spinning body motion with that of a unit mass existing in a central force field. The potential energy $U_1(r)$ may be written,

$$U_1(r) = U(r) + \frac{1}{2}\frac{A^2}{r^2} \tag{2.43}$$

where the potential $U(r)$ is the normalized form of (2.14) and is such that the central force is $F(r) = -\frac{\partial}{\partial r} U(r)$ and,

$$U(r) = \frac{1}{2} ar^2 + \frac{\epsilon}{m+3} r^{m+3} \tag{2.44}$$

and the potential energy due to the centrifugal force is $\frac{1}{2}\frac{A^2}{r^2}$.

Before the orbit equation can be derived, a derivative transformation is required such that,

$$\frac{d}{d\tau} = \frac{A}{r^2}\frac{d}{d\varphi} \quad \text{and} \quad \frac{d^2}{d\tau^2} = \frac{A}{r^2}\frac{d}{d\varphi}\left(\frac{A}{r^2}\frac{d}{d\varphi}\right) \tag{2.45}$$

whence the equation of motion (2. 40) may be written,

$$\frac{A^2}{r^2}\frac{d}{d\varphi}\left(\frac{A}{r^2}\frac{dr}{d\varphi}\right) - \frac{A^2}{r^3} = f(r) \tag{2.46}$$

where the forcing function,

$$f(r) = -ar - \epsilon r^{m+2} \tag{2.47}$$

Introducing a new variable $u = \frac{1}{r}$ equation (2. 46) becomes the orbit equation,

$$\frac{d^2u}{d\varphi^2} + u = \frac{a}{A^2}u^{-3} + \frac{\epsilon}{A^2}u^{-m-4} \tag{2.48}$$

so called because of its analogy with the equation describing the classical orbit problems. Following Goldstein (Ref. 7), rather than attempt to solve (2. 48) formally, use will be made of the energy equation (2. 42) which gives,

$$\dot{r} = \sqrt{2\left[E - U(r)\right] - \frac{A^2}{r^2}} \tag{2.49}$$

and which, with the transformation equation (2. 45) gives,

$$d\varphi = \frac{A\,dr}{r^2\sqrt{2\left[E - U(r)\right] - \frac{A^2}{r^2}}} \tag{2.50}$$

Hence, the integral solution to the orbit equation may be written,

$$\varphi = \varphi_o - \int_{u_o}^{u} \frac{A\,du}{\sqrt{2\left[E - U(u)\right] - A^2u^2}} \tag{2.51}$$

which completes the formal solution to the general non-linear aerodynamic problem under study.

2. The Orbit Integral

Provided the motion is bounded (see Section IV) the path of the nose of the missile lies entirely within the annulus r_1 and r_2. These limiting values for the resultant angle of attack are the turning points and are the roots of the energy equation (2.42). Whence, the resultant angle of attack rotates through an angle of precession whilst r passes from r_1 to r_2 and back again, where

$$\Delta\varphi = 2\int_{r_1}^{r_2} \frac{A\frac{dr}{r^2}}{\sqrt{2\left[E - U(r)\right] - \frac{A^2}{r^2}}} \qquad (2.52)^*$$

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* See Figure 4.

At these turning points r_1 and r_2, $\dot{r} = 0$ but this does not imply a stationary point since angular momentum is conserved and in general $\dot{\varphi} \neq 0$ at these points. The conditions for boundedness are determined from (2. 42) and these form the subject of the next Section. The condition for periodicity in the classical sense, on the other hand, given a bounded solution, requires that $\Delta\varphi$ be a rational function of 2π i. e. that $\Delta\varphi = 2\pi\ p/q$ where p and q are integers. According to Landau and Lifshitz (Ref. 8) the motion would be periodic only for those cases in which the potential energy varies as $1/r$ or as r^2. By induction, it is seen that all non-linear aerodynamic potentials will not yield periodic solutions i. e. the solutions will never repeat themselves. However, if the definition of period is taken to be from peak to peak then values for $\Delta\varphi$ and T (the period) may be obtained by integration and this will be considered in succeeding sections.

The precession $\Delta\varphi$ may be reduced to a more tractable form by a suitable non-dimensionalization. Define a new coordinate,

$$\xi = a^{-\frac{1}{4}} \sqrt{A}\ u \qquad (2.53)^*$$

then the precession reduces to,

$$\Delta\varphi = -2 \int_{\xi_1}^{\xi_2} \frac{\xi^{\frac{m+3}{2}}\, d\xi}{\sqrt{F(\xi\ ; K_1, K_2)}} \qquad (2.54)$$

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* Note that such a coordinate only has meaning for a > 0. In the later discussion on stability such a restriction will be removed.

where ξ_1, ξ_2 are two real roots of $F(\xi; K_1, K_2) = 0$. Expressed in full, the expression under the radical becomes

$$F(\xi; K_1, K_2) \equiv -\xi^{m+5} + K_1 \xi^{m+3} - \xi^{m+1} - K_2 \qquad (2.55)$$

For the bounded motion, the coefficients K_1 and K_2 characterize the system and may be thought of as representing the initial conditions and non-linearity in the system respectively. In terms of the system variables;

$$K_1 = \frac{2E}{\sqrt{a}\ A} \qquad (2.56)$$

$$K_2 = \frac{2\epsilon A^{\frac{1+m}{2}}}{(m+3)\, a^{\frac{m+5}{4}}} \qquad (2.57)$$

With this particular form of polynomial representation for the potential, the general solution for $\Delta\varphi$ may be written

$$\Delta\varphi = -2\int_{\xi_1}^{\xi_2} \frac{\xi^{\frac{m+3}{2}}\, d\xi}{\sqrt{-\xi^{m+5} + K_1 \xi^{m+3} - \xi^{m+1} - K_2}} \qquad (2.58)$$

This will be integrable in terms of simple trigonometrical functions only in certain cases. Goldstein (Ref. 7) discusses the possibilities of the exponent m in the radical to yield solutions expressible in terms of the circular functions and the Legendre elliptic integrals of the first, second and third kinds. Some examples of interest to the aerodynamicist will be considered here.

3. The Orbit Integral Solutions (Examples)

Of especial interest would be those cases where the spinning slender missile was acted upon by a linear restoring moment ($\epsilon = m = 0$), a cubic restoring moment ($m = 1$, $\epsilon > 0$ or < 0) and a quintic restoring moment ($m = 2$, $\epsilon > 0$ or < 0).

(a) The Linear System

For $m = 0$, $\Delta\varphi$ reduces to an elementary integral and may be shown to be $\Delta\varphi = 2\pi$, whence $\Delta\theta \doteq 2\pi - \frac{1}{2} \Lambda P \Delta\tau$. If instead of integrating over the half-cycle, the integration is performed over an arbitrary range of ξ the equation for the resultant angle of attack may be obtained, i.e. write,

$$\varphi = \varphi_o - \int_{\xi_o}^{\xi} \frac{\xi^{\frac{3}{2}} \, d\xi}{\sqrt{F_1(\xi ; K_1)}} \tag{2.59}$$

which upon integration yields the result,

$$2(\varphi_o - \varphi) = \sin^{-1}\left\{\frac{2\eta - K_1}{\sqrt{K_1^2 - 4}}\right\} - \sin^{-1}\left\{\frac{2\eta_o - K_1}{\sqrt{K_1^2 - 4}}\right\} \tag{2.60}$$

where for convenience, $\xi^2 = \eta$. It may be assumed without loss of generality, that $\varphi_o = 0$ when $\xi = \xi_o$. At this point $\dot{\xi} = \dot{r} = 0$ and from the energy relation (2.49)

$$\eta_o = \frac{1}{2}\left[K_1 + \sqrt{K_1^2 - 4}\right] \tag{2.61}$$

when expressed in normalized form. With this value for η_o (2.60) becomes,

$$\sin\left(\frac{\pi}{2} - 2\varphi\right) = \frac{2\eta - K_1}{\sqrt{K_1^2 - 4}} \tag{2.62}$$

and finally in usable form as,

$$\frac{1}{r^2} = \sqrt{\frac{r}{2}}\ \frac{K_1}{A}\left[1 + \frac{\sqrt{K_1^2 - 4}}{K_1}\ \cos 2\varphi\right] \tag{2.63}$$

which will be recognized as the equation of a conic section in the $(r^2, 2\varphi)$ plane. In the (r, θ) or (α, β) plane the solution would exhibit a precession that included the impressed spin rate P. A full discussion of the initial value and bounded solution problems will be reserved to the later sections. Here it is sufficient to show that the familiar linear system is but a special case of the integral solution (2.54) and further to show the elegant form of solution employing this technique.

(b) Cubic Moment

A case frequently encountered in free-flight motions of spinning aerodynamic problems is one where the static restoring moment varies according as the cube of the total angle of attack, i.e.

$$M(r) = -ar - \epsilon r^{m+2} \qquad m = 1 \tag{2.64}$$

For m = 1, the precession $\Delta\varphi$, from the general result (2.54) may be written,

$$\Delta\varphi = \int_{\eta_1}^{\eta_2} \frac{\sqrt{\eta}\, d\eta}{\sqrt{-\eta^3 + K_1\eta^2 - \eta - K_2}} \quad ; \quad \eta = \xi^2 \tag{2.65}$$

The question of boundedness of $\Delta\varphi$ will be given in Section III. If it is assumed that the motion is bounded i. e. that $\Delta\varphi$ remain real then η is bounded such that,

$$F(\eta \,;\, K_1, K_2) = \eta^3 - K_1\eta^2 + \eta + K_2 < 0 \tag{2.66}$$

Assume that $F(\eta \,;\, K_1, K_2)$ can be factored in terms of the ordered three roots $\eta_1 > \eta_2 > \eta_3$ then,

$$\Delta\varphi = \int_{\eta_1}^{\eta_2} \frac{\sqrt{s}\; ds}{\sqrt{(s - \eta_3)(s - \eta_2)(\eta_1 - s)}} \tag{2.67}$$

which may be recognized as a complete elliptic integral of the third kind and from Ref. 8 may be shown to have the solution,

$$\Delta\varphi = \frac{2\sqrt{\eta_2}}{\sqrt{\eta_1 - \eta_2}} \left\{ K + \frac{\pi(a^2 - K^2)\left[1 - \Lambda_0(\vartheta, k)\right]}{2\sqrt{a^2(1 - a^2)(a^2 - k^2)}} \right\} \tag{2.68}$$

where

$$a^2 = \frac{\eta_1 - \eta_2}{\eta_1 - \eta_3} \qquad \text{and} \qquad k^2 = \frac{\eta_3}{\eta_2}\, a^2$$

is the complete elliptic integral of the first kind expressible as,

$$K = \frac{\pi}{2}\left[1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \ldots\right] \tag{2.69}$$

and $\Lambda_o(\vartheta, k)$ is Heumann's Lambda Function

$$\Lambda_o(\vartheta, k) = \frac{\pi}{2}\left[EF(\vartheta, k') + KE(\vartheta, k') - KF(\vartheta, k')\right] \tag{2.70}$$

The argument ϑ is related to the roots of $F(\eta\,;\,K_1, K_2) = 0$ by,

$$\sin\vartheta_s = \sqrt{\frac{1 - a^2}{1 - k^2}} \tag{2.71}$$

In this exact form it is seen that the solution is not quite so elegant as that for the linear system and is certainly less amenable to the data analyst who wishes to interpret a given bounded solution to extract the pertinent motion parameters. A numerical set of computations was performed on the IBM 7090 Computer at Ames to indicate the effect of a cubic non-linearity on the precession $\Delta\varphi$ and the bounds on the motion (determined by the energy level). The results are plotted on Fig. 5. From such a plot it is noted that for $\epsilon < 1$, the precession $\Delta\varphi$ is strongly dependent upon ϵ but for $\epsilon > 1$ the dependence is slight implying that data reduction techniques employing the measurement of $\Delta\varphi$ to deduce the magnitude of the non-linearity are most accurate for $\epsilon < 0\ (1)$.

III. THE INITIAL VALUE PROBLEM

In general, no explicit solutions exist for non-linear differential equations. It is fortunate, however, that by use of well-developed methods, considerable insight may be gained as to the nature of the solutions without recourse to actually requiring the solution. One such method is the phase plane method developed by Poincaré, Liénard and others (Refs. 10 and 11), another method employs the potential energy plane. The equations developed here, thusfar, are particularly amenable to such methods. Other methods provide parallel results and also results not conveniently obtainable by the phase plane method. Such methods encompass perturbation techniques and orbital mechanics. Each method will be presented as required to extract the desired stability criteria and related topics.

A. Phase Plane

For the non-dissipative case ($\delta = 0$) the orbit equation for the general non-linear system was found to be,

$$\frac{d^2u}{d\varphi^2} + u = \frac{a}{A^2} u^{-3} + \frac{\varepsilon}{A^2} u^{-m-4} \qquad (3.1)$$

With the transformation to a new coordinate system (ξ, φ) where,

$$\xi = \sqrt{A}\ u \qquad (3.2)$$

Note: This coordinate ξ is the same coordinate as given in (2.53) except that now for the discussion of stability the restriction $a > 0$

has been removed. In these coordinates, the orbit equation assumes the form,

$$\frac{d^2\xi}{d\varphi^2} + \xi = a\,\xi^{-3} + b\,\xi^{-(m+4)} \qquad (3.3)$$

where the aerodynamic parameters are given as

$$a = -(-1)^j + \frac{1}{4}(1-K)^2 P^2 \qquad (3.4)$$

$$b = \epsilon A^{\frac{1+m}{2}} \qquad (3.5)$$

The quantity (a) is a purely aerodynamic property and does not depend upon the initial conditions. The quantity (b), on the other hand is a function both of the non-linearity in the system and the initial conditions, through the angular momentum A. The initial conditions are u_o, v_o and A in the phase plane, which are directly related to the initial conditions r_o, $\dot{r}_o$, φ_o and $\dot{\varphi}_o$ in the physical plane.

The differential equation for the integral curves is,

$$\frac{dv}{d\xi} = \frac{-\xi + a\xi^{-3} + b\xi^{-m-4}}{v} \qquad (3.6)$$

where the velocity $v = d\xi/d\varphi$. Equation (3.6) is but a special example of the general non-linear conservative system governed by the equation,

$$\frac{dv}{d\xi} = \frac{-f(\xi)}{v} \qquad (3.7)$$

which shows that the integral curves have a zero slope at the points ξ_i, the roots of $f(\xi) = 0$, provided $v \neq 0$ and moreover all the singular points occur along the ξ -axis. (3. 6) may be integrated directly to give,

$$E = \frac{1}{2} v^2 + U(\xi) \tag{3.8}$$

where the potential energy is,

$$U(\xi) = -\frac{1}{2}\xi^2 - \frac{1}{2} a \xi^{-2} - \frac{b}{m+3}\xi^{-m-3} \tag{3.9}$$

The relation between $U(\xi)$ and $f(\xi)$ is,

$$U(\xi) = -\int_{\xi_o}^{\xi} f(\xi)\, d\xi \tag{3.10}$$

As shown by Poincaré (Ref. 13) the only possible singular points to (3. 7) are either centers, saddle points or their confluence. The location of the singular points and their character will provide the desired results for the non-linear aerodynamic system. Since the resultant angle of attack $r = \sqrt{\alpha^2 + \beta^2}$ is always a real quantity greater than zero then only the positive quadrant in the (v, ξ) phase plane need be considered for conditions of stability. Furthermore, a trajectory will exist in the phase plane for all initial conditions such that $E - U(\xi) > 0$. Since the problem requires the solution of a non-linear system with the singular points characterized by centers or saddle-points the linear approximations in the phase-plane method are not valid (Ref. 11) and the singular points are determined from

$$v = 0 \tag{3.11a}$$ *

$$\frac{\partial U(\xi)}{\partial \xi} = 0 \tag{3.11b}$$

describing the conditions for equilibrium. The exact trajectories are analytic and described by,

$$\xi^{m+5} + (v^2 - E)\xi^{m+3} - a\xi^{m+1} - \frac{2b}{m+3} = 0 \tag{3.12}$$

where the (normalized) total energy E can be expressed in terms of the coordinates, viz:

$$E = \frac{1}{2}\xi_m{}^2 - \frac{1}{2}a\xi_m{}^{-2} - \frac{2b}{m+3}\xi_m{}^{-m-3} \tag{3.13}$$

the subscript m denotes the maximum value and ξ_m is the greater of the turning points of the motion. Following Minorsky (Ref. 12) express $f(\xi)$ in a Taylor's series around the singular point to give

$$f(\xi) = \sum_{n=1}^{\infty} \frac{c_n \Psi^n}{n!} \tag{3.14}$$

where $\Psi = \xi - \xi_s$ and $c_n = f^n(\xi)\Big|_{\xi = \xi_s}$. The potential energy is

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* Such a condition does not necessarily imply a stationary point since $\dot{\varphi} \neq 0$ when $v = 0$.

thus automatically expressed as a Taylor's series and the conservation law (3.9) becomes,

$$E = \frac{v^2}{2} - \sum_{n=1}^{\infty} \frac{c_n \Psi^{n+1}}{(n+1)!} \tag{3.15}$$

The discussion now is reduced to considering the potential energy. If the potential energy $U(\xi)$ is a minimum at the singular point, then the system is locally stable; if it is a maximum there, then the system is locally unstable. The condition of a stationary point for the extrema would result in neutral stability which for all practical purposes may be considered as unstable.

It is informative to consider the first term in the series in (3.15) such that the equation describes an ellipse or hyperbola about the singular point, viz:

$$E = \frac{1}{2} v^2 - \frac{1}{2} c_1 \Psi^2 \tag{3.16}$$

Clearly, the condition for stability depends on the sign of c_1 and from what has gone before, $c_1 = -U''(\xi)\big|_{\xi=\xi_s}$. The conditions $c_1 \gtrless 0$ depend on the sign and magnitude of the aerodynamic parameters a, b and ξ_s. Mathematically it is convenient to study the stability in ξ - variable; graphically it is clearer to present the results in the inverse coordinate, i.e.

$$R = \xi^{-1} = \frac{r}{\sqrt{A}} \tag{3.17}$$

In R-variable the energy equation is,

$$E = \frac{1}{2} v^2 + U(R) \tag{3.18}$$

where $$U(R) = -\frac{1}{2} R^{-2} - \frac{1}{2} aR^2 - \frac{b}{m+3} R^{m+3} \tag{3.19}$$

In the equation for the potential (3.19) the first term is due to the centrifugal forces set up by the rotation in the (r, φ) plane; the remaining two terms constitute the potential due to the restoring moments, linear and non-linear. There will be stable regions and unstable regions in R dependent upon the signs and relative magnitudes of a, b and m; the singular points located within the boundaries will characterize the regions as stable or unstable. These singular points will occur along the $v = 0$ axis at the roots of $U'(R) = 0$ i.e. at the roots of,

$$-bR^{m+5} - aR^4 + 1 = 0 \tag{3.20}$$

Assume that the real roots of (3.20) are R_s then stability is assured for $U''(R_s) < 0$. For $U''(R_s) > 0$ the region is an unstable one (the trajectories in the phase plane are hyperbolas, see (3.16)) and for $U''(R_s) = 0$ neutral stability is experienced which for practical purposes may be considered as unstable. Written more concisely,

$$-3R_s^{-4} - a - b(m+2)R_s^{m+1} \begin{cases} < 0 & \text{stable} \\ = 0 & \text{neutral} \\ > 0 & \text{unstable} \end{cases} \tag{3.21}$$

The various cases may now be discussed with the aid of the potential energy curves and the accompanying phase plane.

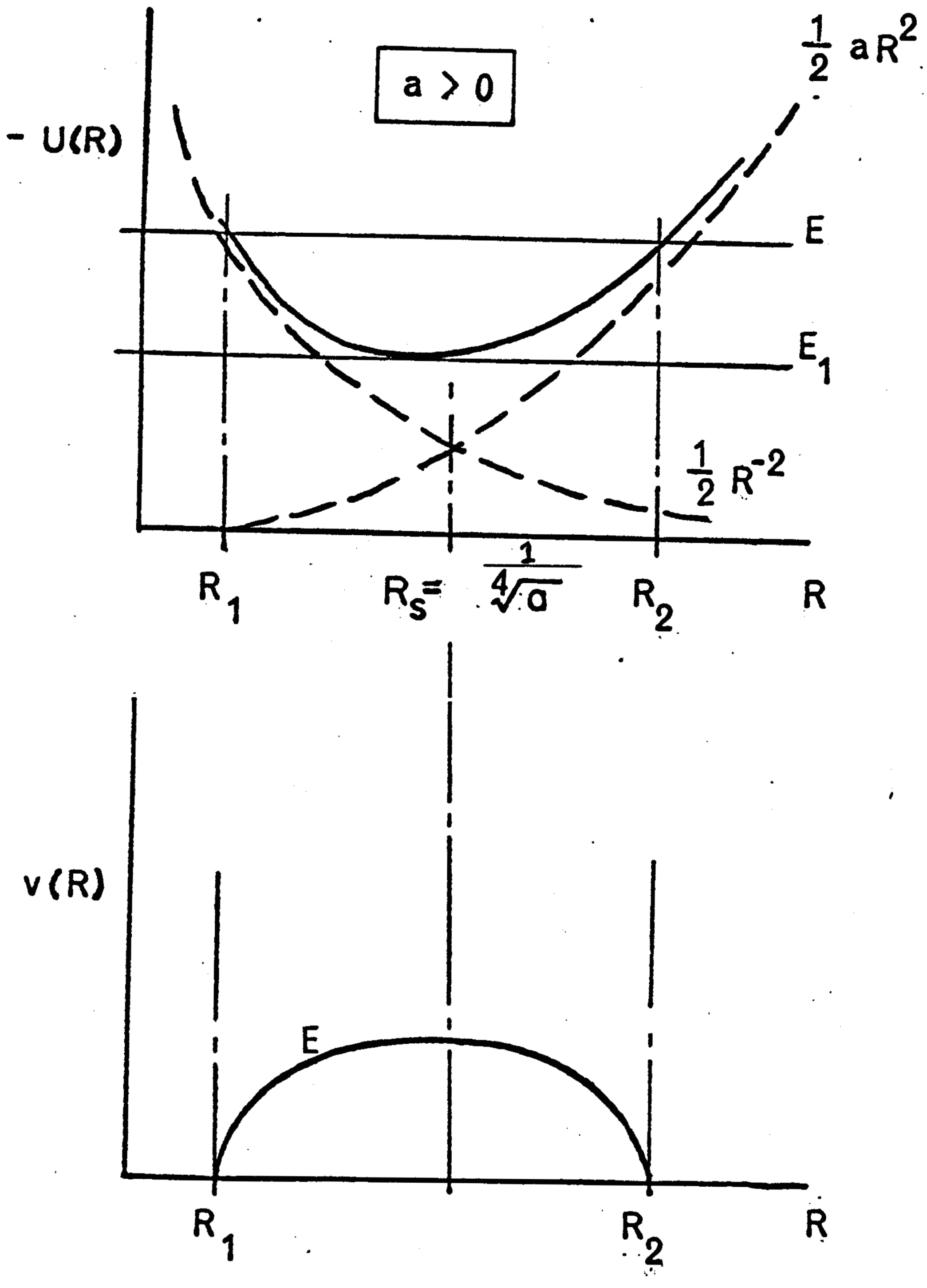
<u>Linear System, $b = 0$</u>

Here the familiar aerodynamic system is described by the potential function

$$U(R) = -\frac{1}{2} R^{-2} - \frac{1}{2} a R^2 \qquad (3.22)$$

The singular points are located at $R_s = \frac{1}{\sqrt[4]{a}}$. For this to be a real point in the physical plane then $a > 0$. Is the region in which R_s is located stable? To answer this question, the criterion $U''(R_s) < 0$ gives $-4a < 0$ which again requires that $a < 0$ for stability. Expressing these results graphically in the complementary potential and phase planes gives Figures (a) and (b). Physical motion is possible provided $E > E_1$. For $E < E_1$ motion is elliptic in (r, φ) plane between the bounds R_1 and R_2 which are the roots of $E - U(R) = 0$, provided only that $a > 0$. For the special case $E = E_1$, circular motion results at the radius $R = \frac{1}{\sqrt[4]{a}}$ where the energy level $E_1 = \sqrt{a}$. Note that the question of stability is unaffected by the magnitude of the angular momentum provided only that $A \neq 0$. This is in contrast to the non-linear system as will be shown. For $a < 0$ the motion is unstable for all R. Consider the full expression for the parameter 'a'

$$a = -(-1)^j + \frac{1}{4}(1 - K)^2 P^2 \qquad (3.23)$$



a > 0
$\frac{1}{2}aR^2$
- U(R)
E
E_1
$\frac{1}{2}R^{-2}$
R_1
$R_S = \frac{1}{\sqrt[4]{a}}$
R_2
R
v(R)
E
R_1
R_2
R

Figure (a)

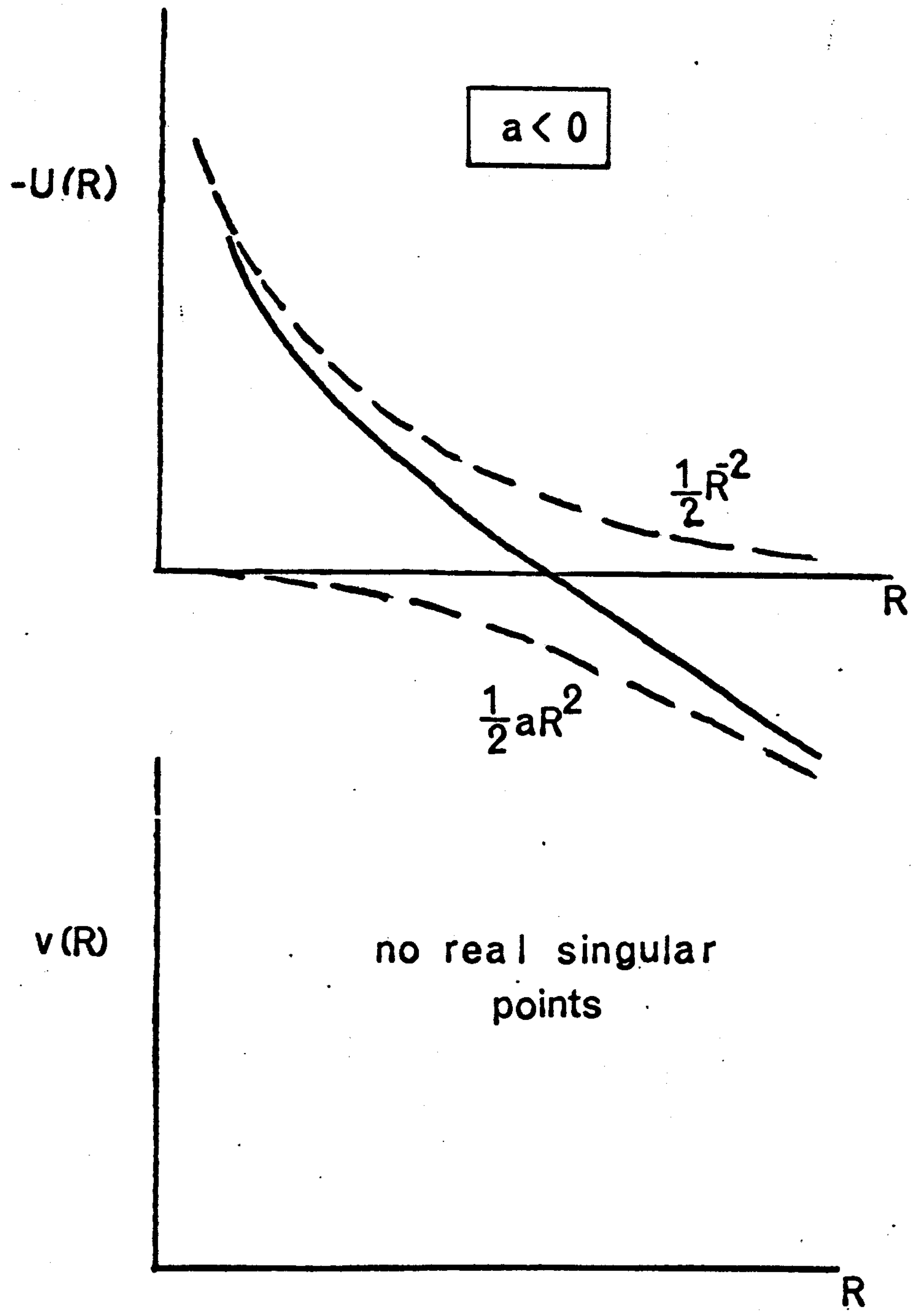


Figure (b)

The question of the sign of 'a' will depend on j and the spin rate P. For the aerodynamically stable case ($j = 1$) $a > 0$ for all P. The aerodynamically unstable case ($j = 0$) however may be spin-stabilized if spun at a rate $P > \sqrt{4/(1 - K)^2}$ i.e. approximately twice the natural frequency of the complementary stable case. This value depends on the inertia characteristics and aerodynamics of the specific vehicle in question. Once determined however, the nature of the motion is completely determined by 'a'.

<u>Non-Linear System, $b \neq 0$</u>

Here the possibilities are a little more varied and the governing cases will be listed.

The singular points will be located along the $v = 0$ axis and at the roots of $U'(R) = 0$, and in general, there will be $\frac{1}{2}(m + 5)$ roots in the R^2 plane of which only the positive roots will have physical significance. The solution of (3.20) to obtain these critical values of angle of attack is left to the usual techniques of numerical analysis in the general case of integer m. For the special case ($m = 1$) of the cubic restoring moment exact solutions are possible.

<u>Example: Cubic Restoring Moment Singular Points</u>

Put $\rho = R^{-(1 + m)}$ then the singular points are located at the roots of,

$$\rho^{\frac{m+5}{m+1}} - a\rho - b = 0 \tag{3.24}$$

For $m = 1$ (3.24) becomes a cubic with the three real roots given by,

$$\rho_{s1} = 2\sqrt{\frac{a}{3}}\ \cos\left\{\frac{1}{3}\cos^{-1}\frac{3\sqrt{3b}}{2a\sqrt{a}}\right\} \tag{3.25a}$$

$$\rho_{s2} = 2\sqrt{\frac{a}{3}}\ \cos\left\{\frac{1}{3}\cos^{-1}\frac{3\sqrt{3b}}{2a\sqrt{a}} + \frac{2\pi}{3}\right\} \tag{3.25b}$$

$$\rho_{s3} = 2\sqrt{\frac{a}{3}}\ \cos\left\{\frac{1}{3}\cos^{-1}\frac{3\sqrt{3b}}{2a\sqrt{a}} + \frac{4\pi}{3}\right\} \tag{3.25c}$$

Certain conditions are implied for the roots to be given by (3.25) which depend on the discriminant $\Delta = (b/2)^2 - (a/3)^3$. For $\Delta < 0$ there will be three real and distinct roots given by (3.25). For $\Delta = 0$ there will be three real roots, two of which will be equal. For $\Delta > 0$ there will exist only one real root, the other two being complex. Only those roots $\rho_{sj} > 0$ have meaning in the present context. There exist four basic possibilities for stable and unstable motion and these will be considered in turn.

Case 1, $a > 0$, $b < 0$

This would be the case of a soft spring acting on a stable linear system. Here the potential function,

$$U(R) = -\frac{1}{2}R^{-2} - \frac{1}{2}aR^2 - \frac{b}{m+3}R^{m+3}$$

The singular points are determined from the roots of (3.24). Since instability is characterized by a local maximum occurring in the potential function, then instability would occur for large R if,

$$\left.\frac{2bR^{m+1}}{(m+3)a}\right|_{R\to\infty} > 1 \qquad (3.26)$$

i. e. if for large R the destabilizing influence of the non-linear term overcame the basically stable linear system. Conversely, $2bR^{m+1}/(m+3)a < 1$ implies a stable system for all R. Expressing these results in the two planes, gives Figures (c) and (d). It is possible then for instability to occur at large R if the non-linearity is large enough to overcome the stabilizing influence of the linear term. The designation 'large R' may not require a large angle of attack for instability but merely a low angular momentum, since as $A \to 0$ then $R \to \infty$ for all r. From this it is seen that nearly planar motions are more susceptible to instability than, say, a nearly circular motion.

The limits of bounded and physical motion are that $E_1 \leqslant E < E_3$. Given a physical motion the upper bound to the stable region is the separatrix in the phase plane with energy level E_3. The equation to this separatrix is given by (3.12) with $E = E_3$ determined from $U''(R_s) < 0$ at the singular point.

<u>Case 2, $a \leqslant 0$, $b < 0$</u>

The case of a soft spring acting on an unstable linear system. Clearly this would be unstable for all R.

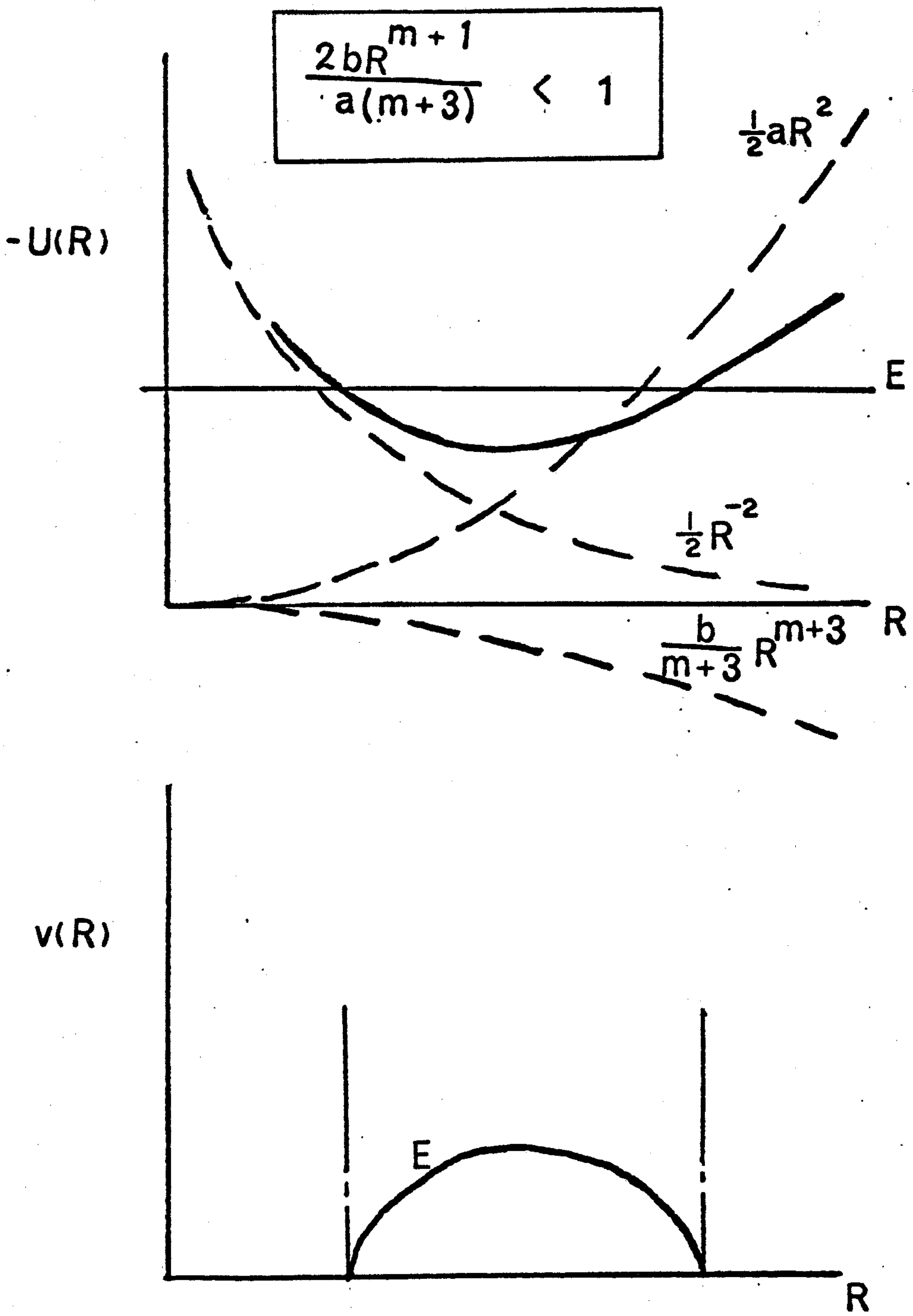


Figure (c)

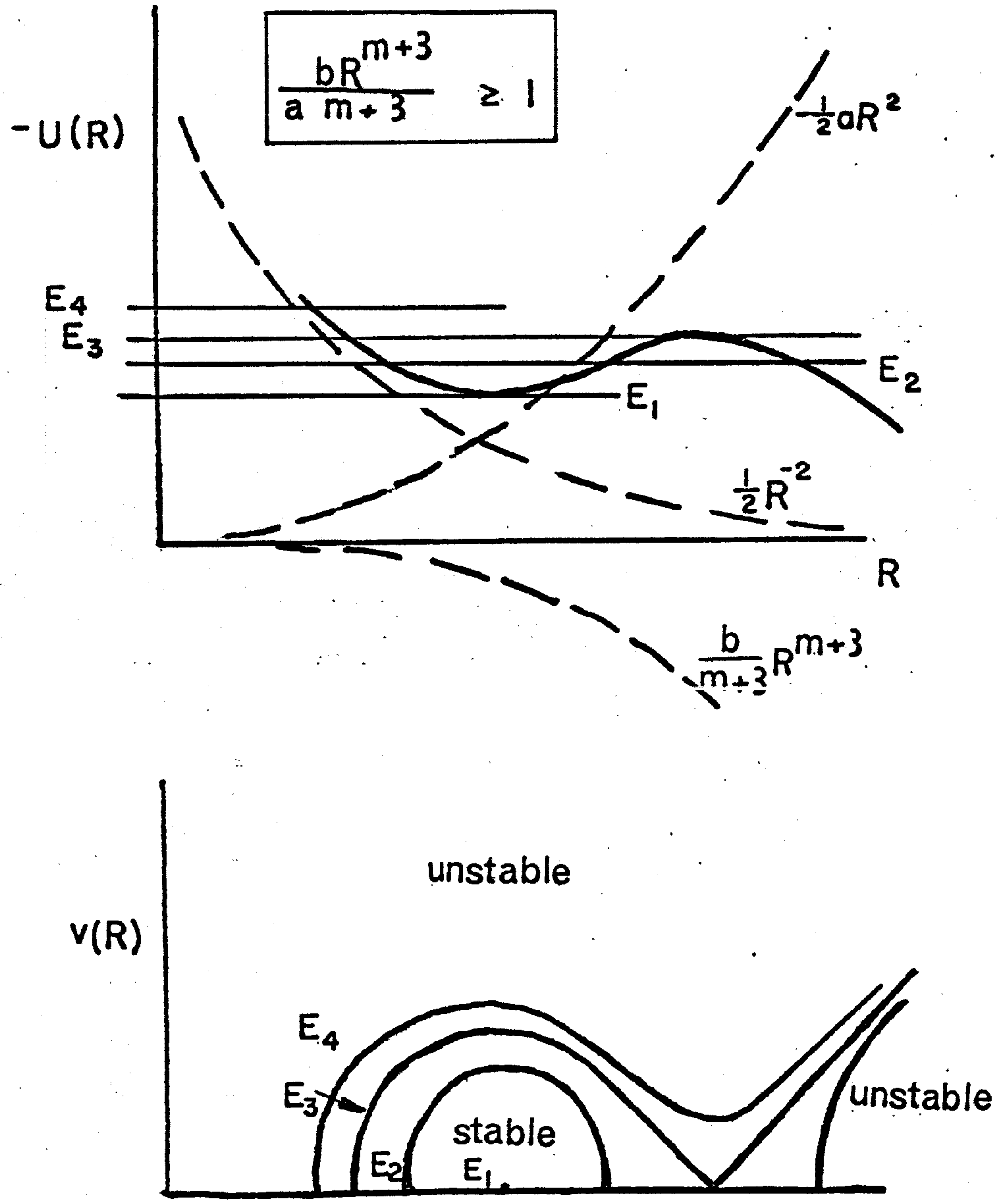

$\frac{bR^{m+3}}{a\ m+3} \geq 1$
-U(R)
$-\frac{1}{2}aR^2$
E_4
E_3
E_2
E_1
$\frac{1}{2}R^{-2}$
R
$\frac{b}{m+3}R^{m+3}$
unstable
v(R)
E_4
E_3
E_2
E_1
stable
unstable

Figure (d)

Case 3, $a \geqslant 0$, $b > 0$

The case of a hard spring acting on a stable linear system. This case would result in stability for all R.

Case 4, $a < 0$, $b > 0$

The case of a hard spring acting on an unstable linear system. This is the complementary case to Case 1 where now the non-linearity is attempting to stabilize an otherwise unstable system. The diagrams are similar (see Figure (e)). The inequalities now require a modulus sign to give the criteria,

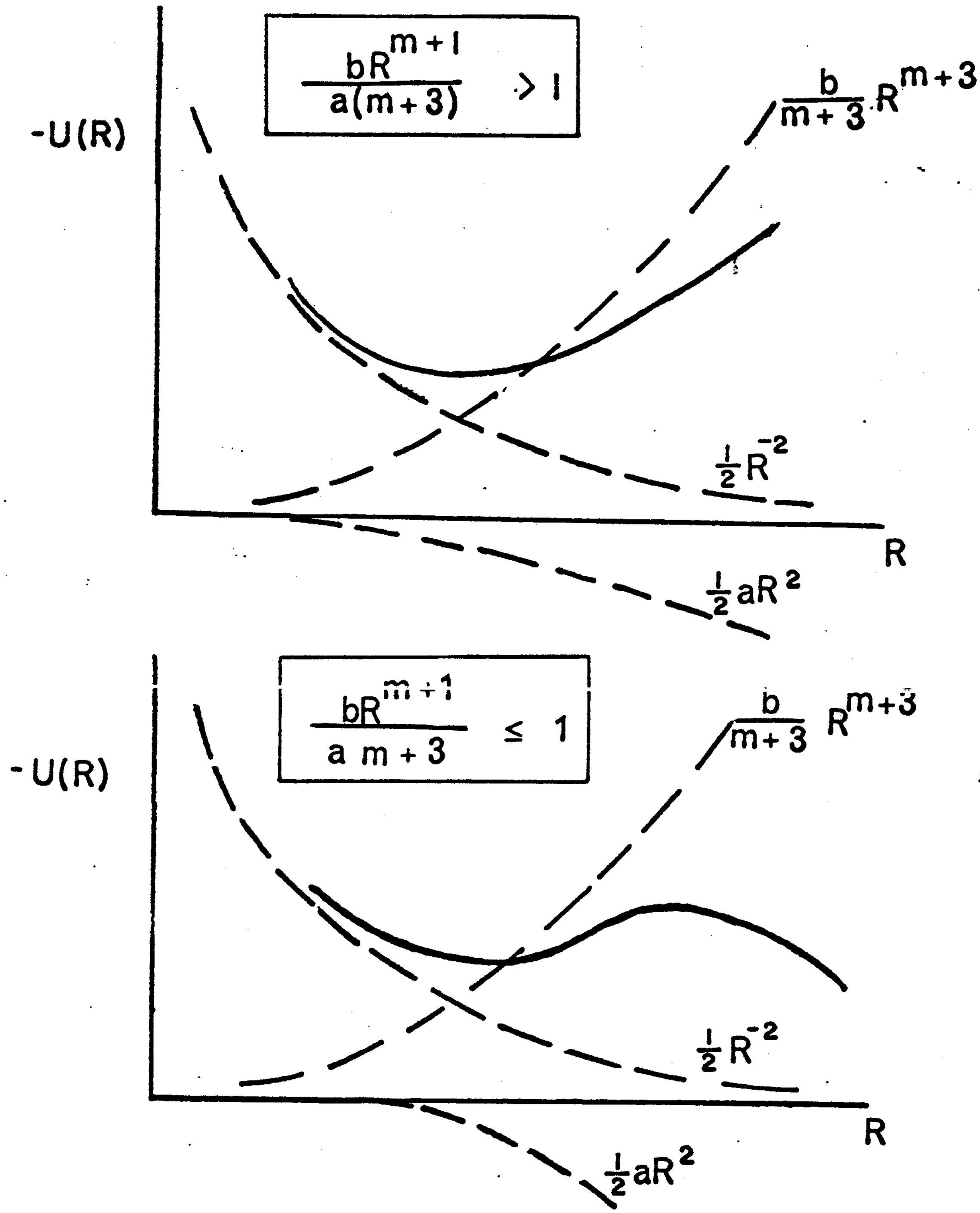
$$\left| \frac{2bR^{m+1}}{a(m+3)} \right| > 1 \quad \text{stable for large R} \qquad (3.27a)$$

$$\left| \frac{2bR^{m+1}}{a(m+3)} \right| \leqslant 1 \quad \text{unstable for large R} \qquad (3.27b)$$

In all of the above cases the initial conditions must be such that physical motion is possible i.e. that $E \geqslant E_1$ where $E_1 = U(R_s)$. The singular points R_s are the real and positive roots of $U'(R) = 0$.

B. Perturbation Method

Having described the stability of the non-linear aerodynamic system and established the necessary bounds it becomes necessary to quantize the solutions for use in extraction of the pertinent parameters. The perturbation theory of Poincaré and later math-



$\frac{bR^{m+1}}{a(m+3)} > 1$
$-U(R)$
$\frac{b}{m+3} R^{m+3}$
$\frac{1}{2} R^{-2}$
R
$\frac{1}{2} aR^{2}$
$\frac{bR^{m+1}}{a\ m+3} \leq 1$
$-U(R)$
$\frac{b}{m+3} R^{m+3}$
$\frac{1}{2} R^{-2}$
R
$\frac{1}{2} aR^{2}$

Figure (e)

ematicians will now be applied to obtain the first order of ϵ solutions to the non-linear problem. No attempt is made at this time to improve the approximations to $0(\epsilon^2)$ as it is not expected that any qualitative change will occur in the solutions.

It has been chosen to concentrate on obtaining the precession $\Delta\varphi$ in what follows, as it is felt that $\Delta\varphi$ is most readily obtainable from a given bounded solution. If the period T of the oscillation is desired, a simple derivative transformation (2.45) applied to the results will yield the desired expression. Consider the normalized orbit equation,

$$\frac{d^2\xi}{d\varphi^2} + \xi = a\xi^{-3} + b\xi^{-m-4} \tag{3.28}$$

where the coefficients (a, b) are given by (3.6). The perturbation method as applied to the physical coordinates $\alpha(\tau)$ and $\beta(\tau)$ did not yield suitable results and the reader is referred to Appendix A for such a derivation. The transformation to a polar coordinate system was made early in the analysis in anticipation of this difficulty and a series solution for $\xi(\varphi)$ will be assumed in the form,

$$\xi(\varphi;\epsilon) = \sum_{n=0}^{N} \epsilon^n \xi_n(\varphi^*) \tag{3.29}$$

where the rotational variable φ has been distorted by a "frequency" factor due to the non-linearity in the system.

$$\varphi^* = (1 + \epsilon\Omega_1 + \ldots\ldots)\varphi \tag{3.30}$$

so that for $\epsilon = 0$ the two rotational scales are identical. A substitution of (3.29) and (3.30) into (3.28) gives,

$$\left(\frac{d^2\xi_o}{d\varphi^{*2}} + \xi_o - a\,\xi_o^{-3}\right) + \epsilon\left(\frac{d^2\xi_1}{d\varphi^{*2}} + \xi_1 - a\,\xi_1^{-3} - b_1\,\xi_o^{-m-4} + 2\,\Omega_1\,\frac{d^2\xi_o}{d\varphi^{*2}}\right) + \epsilon^2(\,.\,.\,.\,.\,.\,.\,.\,.\,) = 0 \qquad (3.31)$$

Equating the coefficients of the like powers of ϵ there derives the following recursive system

$$\frac{d^2\xi_o}{d\varphi^{*2}} + \xi_o - \frac{a}{\xi_o^3} = 0 \qquad (3.32a)$$

$$\frac{d^2\xi_1}{d\varphi^{*2}} + \xi_1 - \frac{a}{\xi_1^3} = +\,b_1\,\xi_o^{-m-4} - 2\,\Omega_1\,\frac{d^2\xi_o}{d\varphi^{*2}} \qquad (3.32b)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\frac{d^2\xi_n}{d\varphi^{*2}} + \xi_n - \frac{a}{\xi_n^3} = F(\xi_{n-1};\, b_1\,;\,\Omega_n,\,\ldots) \qquad (3.32c)$$

with the given initial conditions $\xi_i(0)$ and $\frac{d\,\xi_{i(0)}}{d\,\varphi^*}$. The solution to the zeroth order equation has been found and is,

$$\xi_o^{\,2} = c\left[1 + \frac{\sqrt{c^2 - a}}{c}\,\cos\,2\,\varphi^*\right] \qquad (3.33)$$

which describes a conic section in the $(1/\xi_o^2, 2\varphi^*)$ plane. Again it will be assumed that the conditions are such that the motion is bounded (see Section IV), which means that (3.33) describes either a circle or an ellipse in the $(1/\xi_o^2, 2\varphi^*)$ plane. The straight line or planar motion case is excluded from this analysis by the requirement that $A \neq 0$. From (3.33) the necessary operations for the right-hand side of (3.32b) may be performed, viz:

$$\xi_o^{-(m+4)} = c^{-\left(\frac{m+4}{2}\right)} \left[1 + \sqrt{1 - \frac{a}{c^2}} \cos 2\varphi^*\right]^{-\left(\frac{m+4}{2}\right)} \tag{3.34}$$

$$\xi_o'' = -\sqrt{\frac{c^2 - a}{c}} \left[1 + \sqrt{1 - \frac{a}{c^2}} \cos 2\varphi^*\right]^{-\frac{1}{2}}$$

$$\left\{2 \cos 2\varphi^* + \sqrt{1 - \frac{a}{c^2}} \sin 2\varphi^* \left[1 + \sqrt{1 - \frac{a}{c^2}} \cos 2\varphi^*\right]^{-1}\right\} \tag{3.35}$$

For bounded motion $\sqrt{1 - \frac{a}{c^2}} < 1$ and hence the term $\left(\sqrt{1 - \frac{a}{c^2}}\right) \cdot \cos 2\varphi^* < 1$ and an expansion procedure will provide results of sufficient accuracy to give

$$\xi_o^{-(m+4)} = c^{-\left(\frac{m+4}{2}\right)} \left[1 - \frac{m+4}{2}\sqrt{1 - \frac{a}{c^2}} \cos 2\varphi^* + \dots\right] \tag{3.36}$$

$$\xi_o'' = -2 \sqrt{\frac{c^2 - a}{c}} \cos 2\varphi^* + \dots \tag{3.37}$$

which together with the expanded linear solution

$$\xi_o = \sqrt{c} \left[1 + \tfrac{1}{2}\sqrt{1 - \frac{a}{c^2}} \cos^2\varphi^* + \dots\right] \tag{3.38}$$

will give to $O(\epsilon)$ the equation for the correction to the linear solution

$$\frac{d^2\xi_1}{d\varphi^{*2}} + \xi_1 - \frac{a}{\xi_1^3} = 4\Omega_1\sqrt{\frac{c^2-a}{c}}\cos 2\varphi^* + \sqrt{c}\, b_1$$

$$+ \left(\frac{m+4}{2}\right) b_1 \sqrt{\frac{c^2-a}{c}}\cos 2\varphi^*$$

$$+ \ldots\ldots\ldots \tag{3.39}$$

To eliminate these secular terms then,

$$-4\Omega_1 + \left(\frac{m+4}{2}\right) b_1 = 0 \tag{3.40}$$

From (3.40) and (3.5) with $b = \epsilon b_1$ then the non-linear frequency correction is

$$\Omega_1 = +\frac{m+4}{8} A^{\frac{1+m}{2}} \tag{3.41}$$

So that to $O(\epsilon)$

$$\varphi^* = \left(1 + \frac{m+4}{8}\epsilon A^{\frac{1+m}{2}} + \ldots\right)\varphi \tag{3.42}$$

but $\Delta\varphi = \frac{2\pi}{4}\Delta\Omega$ hence

$$|\Delta\varphi| = \frac{m+4}{32}\epsilon A^{\frac{1+m}{2}} 2\pi \tag{3.43}$$

For the special case $m = 1$ this gives

$$\Delta\varphi = \frac{5}{32}\, \epsilon A\ 2\pi \tag{3.44}$$

A comparison of this result with the exact results* and the expansion of the orbit integral are given in Figure 5 .

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* Obtained by numerical integration applied to constructed examples on an IBM 7090 digital computer.

IV. THE BOUNDED VALUE PROBLEM

Before applying the preceding analysis to the non-linear problem it is informative to reiterate the linear aerodynamic system in this new context. It will be shown that the elegant orbit formulation yields known results in a more direct and compact form and further to give new insight into the spinning aerodynamic body problem. The extraction of the pertinent aerodynamic parameters from a known linear amplitude history will follow by way of a concluding example. The linear aerodynamic spinning body describes an amplitude history according to the exact solution,

$$\xi_o^2 = E\left[1 + \sqrt{1 - \frac{a}{E^2}}\cos 2\varphi\right] \tag{4.1}$$

Whether or not the orbit is bounded will depend on the total energy (initial conditions) and the orbits eccentricity $e = \sqrt{1 - a/E^2}$. The governing relations are,

$e \geq 1$	$E \geq 0$	unbounded motion
$e < 1$	$E < 0$	bounded motion

and in particular, circular motion is possible if $e = 0$.

Unbounded Motion

For $e < 1$ the motion is bounded and for $e > 1$ the motion is unbounded. The transitional state $e = 1$ will determine any tendency to resonate or pass from one state to another. This resonance condition implies $a = 0$ i.e. $-(-1)^j + \frac{1}{4}(1-K)^2P^2 = 0$ which would imply a resonant spin value such that,

$$P_R^{\ 2} = \frac{4(-1)^j}{(1-K)^2} \tag{4.2}$$

or, in terms of the physical aerodynamic properties,

$$p_R^{\ 2} = \frac{4\,|M_o|\,(-1)^j}{(1-K)^2} \tag{4.3}$$

The square of the spin (P_R) is simply a mathematical way of showing that this critical value would occur whether the spin was in the direction of the inherent rotational motion $\theta(t)$ or against it. For this resonant value to exist, however, the right-hand side of (4.3) must be a positive quantity. For $j = 1$ this value does not exist, as found earlier. Indeed this "resonance" value for the spin rate is actually that value for which the spinning body passes from an unstable state to a stable one and the body has become spin-stabilized. A related condition for unbounded motion is that $E \geqslant 0$; again the transitional state $E = 0$ will determine the limiting values for $\xi_o(0)$ for the bounded motion. From (3.13),

$$O = \xi_m^{\ 2} + a\xi_m^{\ -2} \tag{4.4}$$

which states that for unbounded motion $\xi_m^{\ 2} > \sqrt{a}$ a known condition from considerations in the potential energy plane.

Bounded Motion

Bounded motion for the linear case is assured if $e < 1$ and $E < 0$. That the total energy be negative for bounded motion is a known result and is automatically satisfied for all real ξ_m as seen from (3.13), on placing $b = 0$ provided $\xi_m^{\;2} < \sqrt{a}$ i.e.

$$E = -\frac{1}{2}\left[\xi_m^{\;2} + a\,\xi_m^{\;-2}\right] \qquad (4.5)$$

In general, the bounded motion will be elliptic when traced in the proper coordinate system, viz: in the ($\xi_o^{\;-2}$, 2φ) plane (see equation (4.1)).

Circular Motion

A special case of the bounded motion solution is when the "force" derived from the potential $U(\xi)$ just balances the centrifugal force set up by the rotation such that the angle of attack is a constant i.e. circular motion. The eccentricity of such an orbit is zero ($e = 0$) which would arise when $E = -\sqrt{a}$, whence circular motion would result when the spin rate is given by,

$$P_c^{\;2} = \frac{-(-1)^j + E^2}{\frac{1}{4}(1 - K)^2} \qquad (4.6)$$

From (4.2) and (4.6) it is seen that

$$P_c^{\;2} = P_R^{\;2} + \frac{4E^2}{(1 - K)^2} \qquad (4.7)$$

and on substitution of the aerodynamic parameters it can be shown that the spin rate required for circular motion is approximately one-half that for spin stabilization.

To complete the discussion for the linear aerodynamic problem use will be made of the orbit integral solution to extract the static stability parameter from a given bounded solution.

A. Linear System

The orbit integral (2.54) provides the simplest result for the linear system when transformed back to the time variable, with the aid of (2.45) to give the simple result (for $\epsilon = 0$),

$$\sqrt{a} = \frac{-\pi}{\Delta\tau} \tag{4.8}$$

Expressed in terms of physical variables this gives,

$$C_{m_\alpha} = -\left[\left(\frac{\pi}{\Delta t}\right)^2 + \frac{1}{4}(1-K)^2 p^2\right] \frac{2I}{\rho U^2 S \ell} \tag{4.9}$$

If it is found that the affect of the spin is small then the following simple relation holds,

$$C_{m_\alpha} = -\left(\frac{\pi}{\Delta s}\right)^2 \frac{2I}{\rho S \ell} \tag{4.10}$$

where $\Delta s = U\Delta t$ is the distance flown along the flight path during one complete oscillation of the angle of attack. Hence knowing the number of peaks in the amplitude α, β plane and the distance flown (or time taken) for each peak the static stability parameter (C_{m_α}) is easily extracted using (4.10). If the roll rate is measurable from flight data then (4.9) should be used. Figure 6 illustrates how the required information is obtained from a given flight trajectory (taken from Ref. 14).

B. Non-Linear System

Once the assumption of non-linearity in the system is accepted the number of possible solutions becomes infinite as opposed to a finite number (enumerated above) for the linear system. It would be desirous in this instance to have a method whereby the data itself indicates the magnitude and degree of non-linearity in the system to the analyst. The alternative is to assume the form of the non-linearity a priori and force the data to conform (say fit a cubic dependence for the pitching or restoring moment). It is possible, by such methods to obtain "good fits" for the data in hand but it is questionable whether or not one can predict the stability characteristics of other missiles within the same family as the one analyzed. It is clear that the first approach is the better although more difficult to apply but to this end the following experiment is suggested.

Suggested Experiment

From the general result, the precession in the α, β plane of the resultant angle of attack loops is given by,

$$\Delta\varphi = \frac{m + 4}{32} \; 2\pi\epsilon \; A^{1 + m} \tag{4.11}$$

Taking logarithms of both sides gives,

$$\ln \Delta\varphi = \ln \frac{m + 4}{32} \, 2\pi\epsilon \quad + \; (1 + m) \ln A \tag{4.12}$$

If now several launchings or free-flight tests are made on the configuration under study where various rates of spin are intentionally given, a series of results ($\Delta\varphi$) will be obtained for a range of angular momentum (A). The angular momentum may be deduced from the data by making use of the sectorial velocity which from Kepler's second law of orbital motion is one-half the angular momentum (see Figure 7). The angular momentum may then be computed for the particular free-flight data under analysis together with the precession $\Delta\varphi$. Plotting these quantities on a logarithm plot will enable the magnitude and degree of non-linearity to be deduced from the data. Figure 7 serves to illustrate the data reduction technique.

V. CONCLUDING REMARKS

The foregoing analysis has shown that the many isolated cases of spinning aerodynamic body behaviour are but special examples of a general theory presented here. Further it has been shown that the aerodynamic problem may be analogized with the orbital motions encountered in physics with all their elegant solutions and general results. The behaviour of the spinning aerodynamic body in a non-linear field is governed by a potential which has been studied with a view to establishing stability boundaries for the various families of aerodynamic bodies (encompassing such characteristics as spin rate, inertial distribution, form of static restoring moment, etc.). The analysis has proceeded further to study the bounded solutions to yield easy-to-apply methods for parameter extraction. Methods are given for extracting the parameters from linear data (together with an example) and from non-linear data. In the case of the non-linear system emphasis has been placed on allowing the data to indicate the magnitude and degree of non-linearity in the system rather than fit assumed forms to the data.

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APPENDIX A

The Non-Existence of Periodic Solutions for Physical Coordinates : $\alpha(\tau)$, $\beta(\tau)$

To investigate the solution for the physical coordinates $\alpha(\tau)$, $\beta(\tau)$ consider the non-dissipative case, $\delta = 0$, such that the equations (2.22) become,

$$\beta'' + \beta' - (1 + K)\, P\alpha' + K_1 P\alpha - \epsilon\beta\,(\alpha^2 + \beta^2)^{\frac{1+m}{2}} = 0 \qquad \text{(A. 1)}^*$$

$$\alpha'' + \alpha' + (1 + K)\, P\beta' - K_1 P\beta - \epsilon\alpha\,(\alpha^2 + \beta^2)^{\frac{1+m}{2}} = 0 \qquad \text{(A. 2)}^*$$

To describe the system completely two characteristic time scales would be required; one to describe the non-linearity (ϵ) and the other to describe the effect of spin (P), on the solution. Since it is not clear that $\alpha(\tau)$ and $\beta(\tau)$ are periodic even for $P = 0$, this case will be considered first to illustrate the solution. This obviates the need for two time scales and solutions are assumed to be of the form,

$$\beta(\tau;\epsilon) = \sum_{n=0}^{N} \epsilon^n \beta_n(t^*) \qquad \text{(A. 3a)}$$

$$\alpha(\tau;\epsilon) = \sum_{n=0}^{N} \epsilon^n \alpha_n(t^*) \qquad \text{(A. 3b)}$$

---

* For this section it has been found convenient to distort the time scale such that $- \;{}^2 = 1$

where $$t^* = \omega\Upsilon = (1 + \epsilon\omega_1 + \dots)\Upsilon \qquad \text{(A. 3c)}$$

and ω is the natural frequency (if it exists) of the non-linear oscillation and has yet to be determined. Substitution of (A. 3) into (A. 1) and (A. 2) gives,

$$\beta_o''\left[1 + 2\epsilon\omega_1 + \dots\right] + \epsilon\beta_1''\left[1 + \dots\right] + \beta_o + \epsilon\beta_1 + \dots$$
$$- \epsilon\left[\beta_o + \epsilon\beta_1 + \dots\right]\left[a_o^2 + 2\epsilon a_1 + \dots + \beta_o^2 + 2\epsilon\beta_1 + \dots\right]^{\frac{1+m}{2}} = 0$$

(A. 4a)

$$a_o''\left[1 + 2\epsilon\omega_1 + \dots\right] + \epsilon a_1''\left[1 + \dots\right] + a_o + \epsilon a_1 + \dots$$
$$- \epsilon\left[a_o + \epsilon a_1 + \dots\right]\left[a_o^2 + 2\epsilon a_1 + \dots + \beta_o^2 + 2\epsilon\beta_1 + \dots\right]^{\frac{1+m}{2}} = 0$$

(A. 4b)

whence derives the recursive system

$$\beta_o'' + \beta_o = 0 \qquad \text{(A. 5a)}$$

$$a_o'' + a_o = 0 \qquad \text{(A. 5b)}$$

$$\beta_1'' + \beta_1 = -2\,\omega_1\beta_o'' + \beta_o a_o^2 + \beta_o^3 \qquad \text{(A. 6a)}$$

$$a_1'' + a_1 = -2\,\omega_1 a_o'' + a_o\beta_o^2 + a_o^3 \qquad \text{(A. 6b)}$$

. . . . .

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$$\beta_n'' + \beta_n = F(\beta''_{n-1}, \dots\dots, \omega_{n-1}) \qquad \text{(A. 7a)}$$

$$a_n'' + a_n = G(\beta''_{n-1}, \dots\dots, \omega_{n-1}) \qquad \text{(A. 7b)}$$

where for the purposes of algebraic simplicity the case $m = 1$ has been chosen. Now, from (2.34b) it is seen that for $P = 0$, the angular momentum A may be written,

$$A = \begin{vmatrix} \eta_1 & \dot{\eta}_1 \\ \eta_2 & \dot{\eta}_2 \end{vmatrix} \tag{A.8}$$

where $\eta_1 = \beta(\tau)$ and $\eta_2 = \alpha(\tau)$. Provided the initial conditions are antisymmetric i.e. $\eta_i(0) = a_{ij}\,\delta_{ij}\,\dot{\eta}_j$ where the Kronecker delta $\delta_{ij} = 0$, $i \neq j$ and $\delta_{ij} = 1$, $i = j$ and a_{ij} is an arbitrary scale factor then $A \neq 0$ and the motion is non-planar. From these conditions, assume that the zeroth order of ϵ solutions are of the form,

$$\alpha_o = A_o \cos t^* \tag{A.9a}$$

$$\beta_o = B_o \sin (t^* + \phi) \tag{A.9b}$$

where it is noted that $\phi = \pi/2$ implies planar motion ($A = 0$) and is to be excluded. After some algebra the first order of ϵ solutions become,

$$\begin{aligned}\beta_1'' + \beta_1 = {} & 2\,\omega_1 B_o \left[\sin t^* \cos\phi + \cos t^* \sin\phi\right] + \frac{3}{4}\, A_o^{\,2}\, B_o \\ & \left[\sin t^* \cos\phi + \cos t^* \sin\phi\right] + B_o^{\,3}\left[\frac{3}{4}\,\sin t^* \cos^3\phi \right. \\ & + \frac{3}{4}\sin t^* \cos\phi\,\sin^2\phi + \frac{1}{4}\cos t^* \cos\phi\,\sin 2\phi \\ & + \frac{1}{4}\,\cos t^* \sin\phi\,\cos^2\phi + \frac{3}{4}\cos t^* \sin^3\phi + \frac{1}{4}\sin t^* \\ & \left.\sin\phi\,\sin 2\phi\right] + \text{non-secular terms}\end{aligned} \tag{A.10a}$$

and

$$a_1'' + a_1 = 2\,\omega_1 A_o \cos t^* + A_o B_o^{\,2} \left[\frac{1}{4} \cos t^* \cos^2 \phi + \frac{3}{4} \cos t^* \right.$$

$$\left. \sin^2 \phi + \frac{3}{4} \sin t^* \sin 2\phi \right] + A_o^{\,3}\, \frac{3}{4} \cos t^*$$

$$+ \text{ non-secular terms} \tag{A. 10b}$$

To avoid secular terms which would invalidate the assumption of a convergent series solution (A. 3), the integration constants A_o, B_o must satisfy the equations:

$$(2\,\omega_1 B_o + \frac{3}{4} A_o^{\,2} B_o) \cos \phi + \frac{3}{4} B_o^{\,3} \cos^3 \phi + \frac{3}{4} B_o^{\,3} \cos \phi \sin^2 \phi$$

$$+ \frac{1}{4} \sin \phi \sin 2\,\phi = 0 \tag{A. 11}$$

$$(2\,\omega_1 B_o + A_o^{\,2} B_o) \sin \phi + \frac{1}{4} B_o^{\,3} \cos \phi \sin 2\,\phi + \frac{1}{4} B_o^{\,3} \sin \phi \cos^2 \phi$$

$$+ \frac{3}{4} \sin^3 \phi = 0 \tag{A. 12}$$

$$2\,\omega_1 A_o + \frac{1}{4} A_o B_o^{\,2} \cos^2 \phi + \frac{3}{4} A_o B_o \sin^2 \phi + \frac{3}{4}\, A_o^{\,3} = 0 \tag{A. 13}$$

$$\frac{3}{4}\, A_o B_o^{\,2} \sin 2\,\phi = 0 \tag{A. 14}$$

Since $A \neq 0$ in this general non-planar case then (A. 14) requires that $A_o \neq 0$, $B_o \neq 0$ and thus the phase angle ϕ must have one of the following values

$$\phi = 0\,,\ \frac{\pi}{2}\,,\ \pi\,,\ \frac{3\pi}{2}\,,\ \ldots$$

But $\phi = 0$ implies $B_o = 0$ for ω_1 to exist which violates $A \neq 0$. Also $\phi = \pi/2$ implies $A = 0$. Substitution of $\phi = \pi$ into (A. 11) and (A. 12) will reveal once more that $B_o = 0$. From induction therefore it is seen that there exist no conditions such that the non-planar motion is periodic and the method fails in this coordinate system.

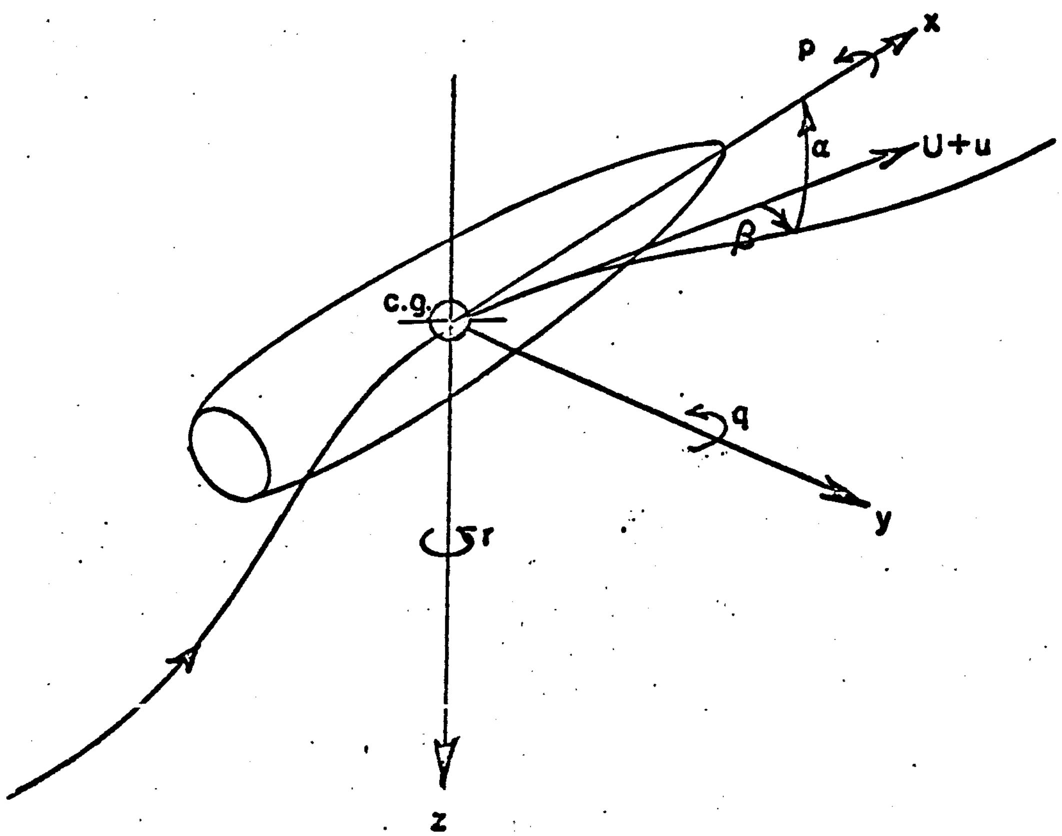


FIG. I NOMENCLATURE

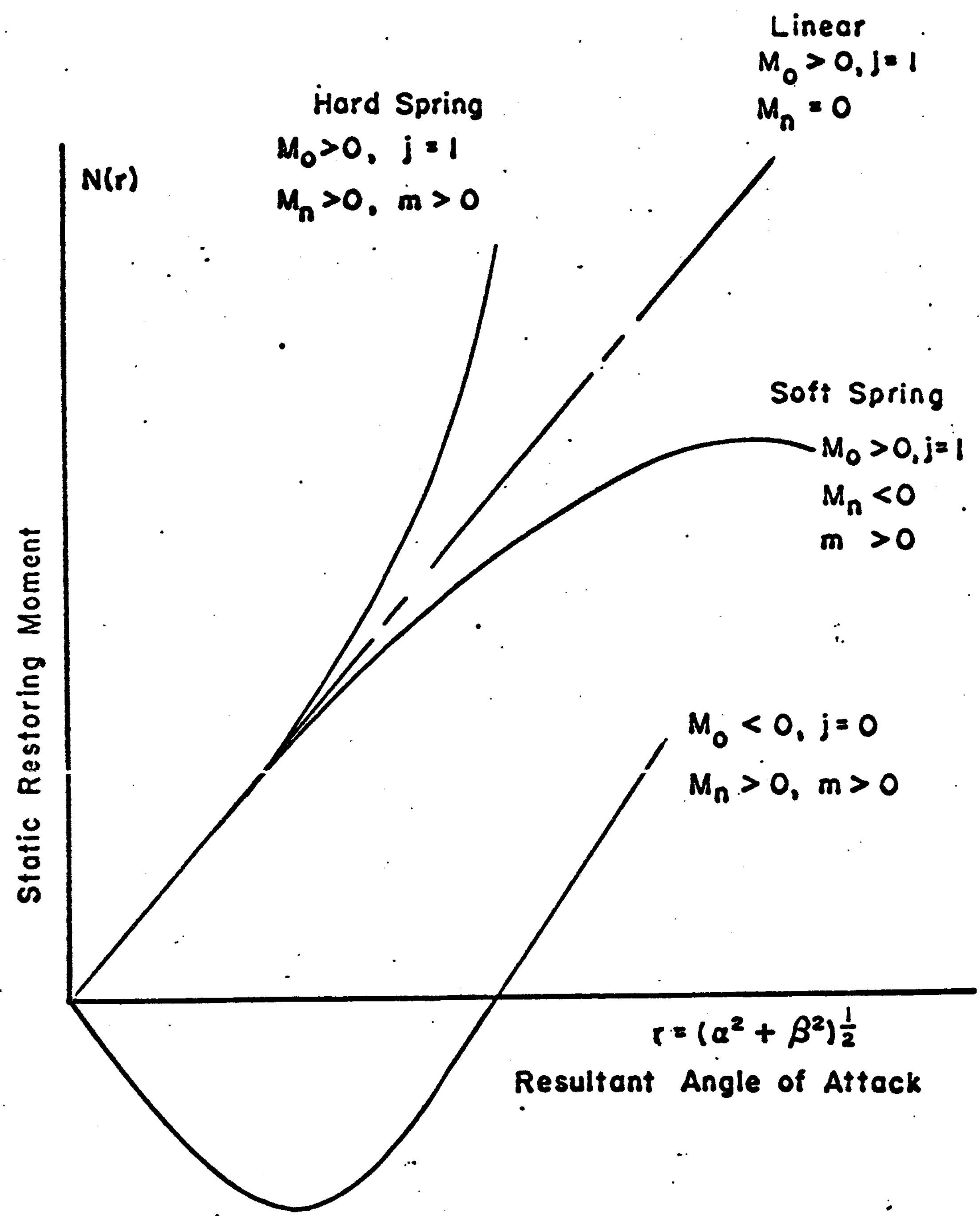


FIG. 2 STATIC RESTORING MOMENT REPRESENTATION

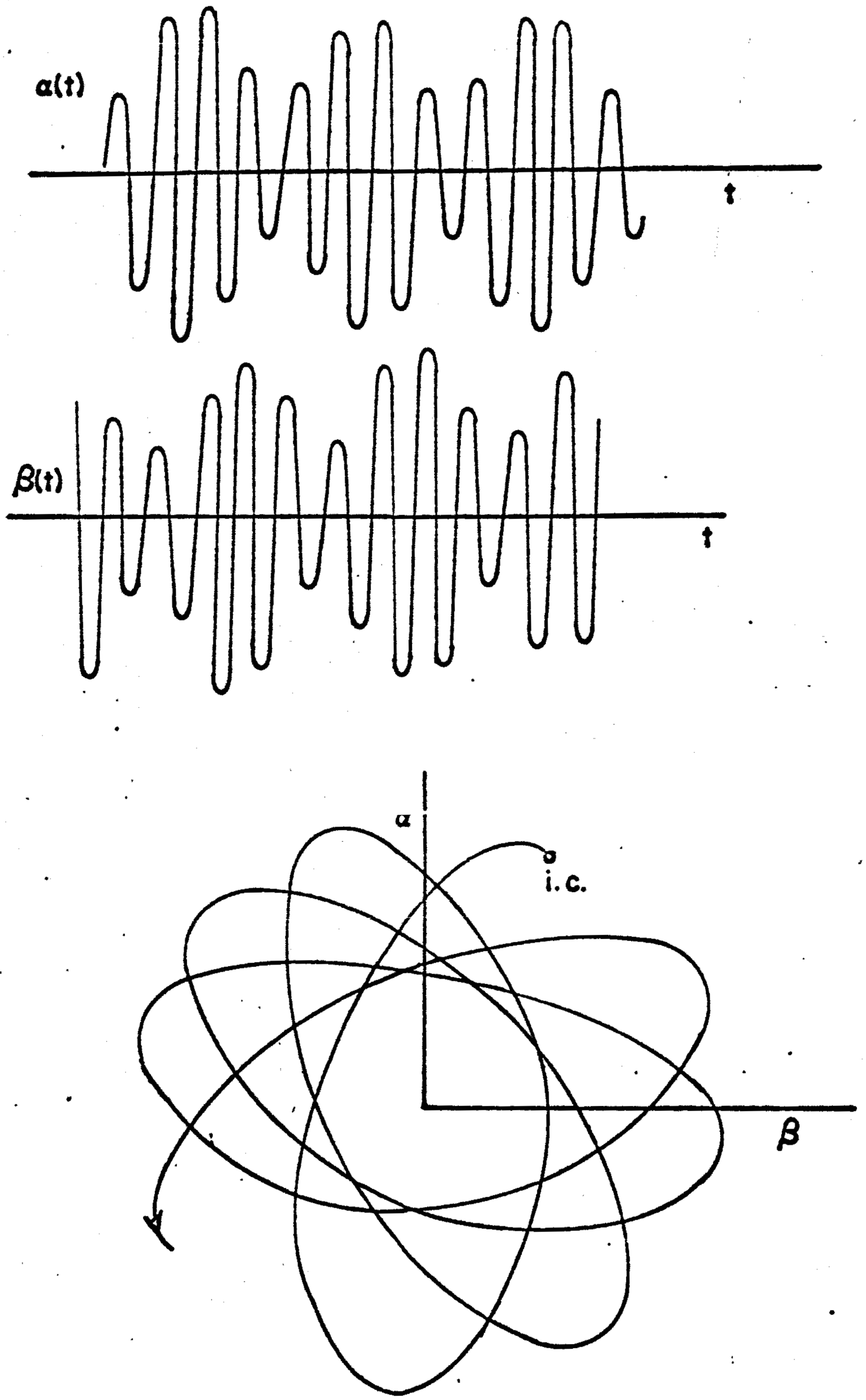


FIG. 3 PHYSICALLY OBSERVED MOTION

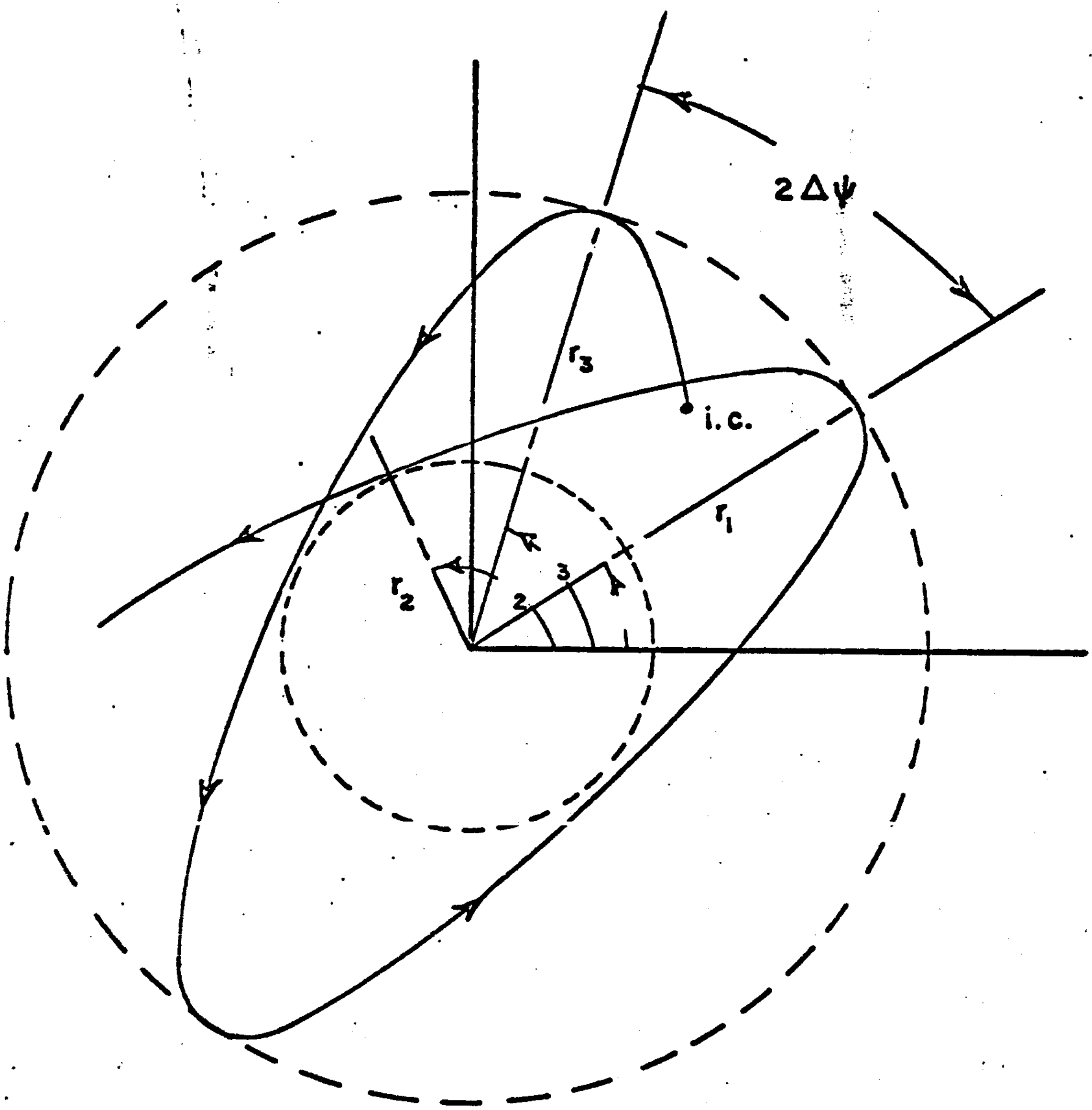


FIG. 4 OBSERVABLE BOUNDED MOTION PARAMETERS

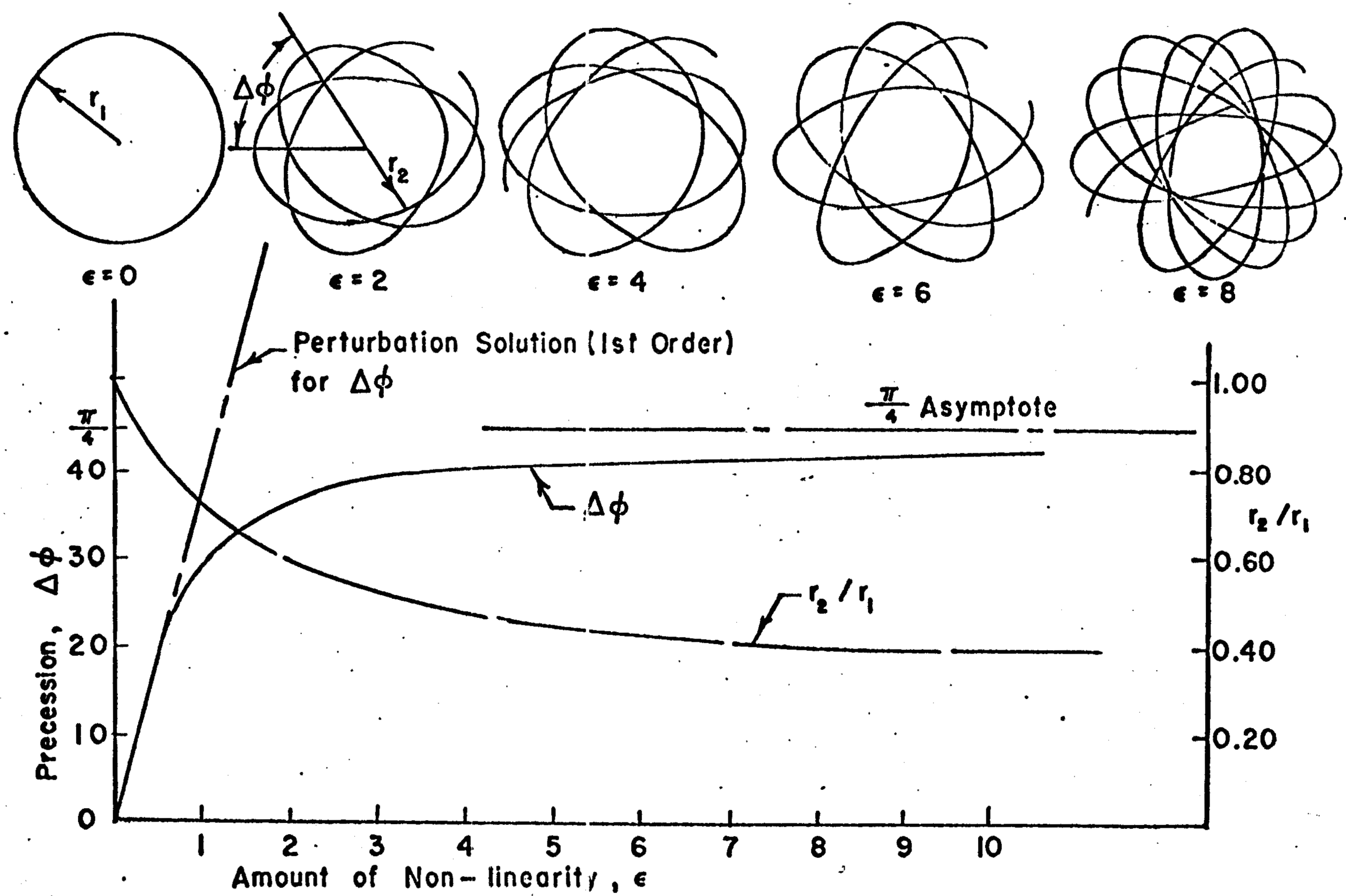


FIG. 5 PRECESSION DUE TO VARIOUS AMOUNTS OF CUBIC NON-LINEARITY

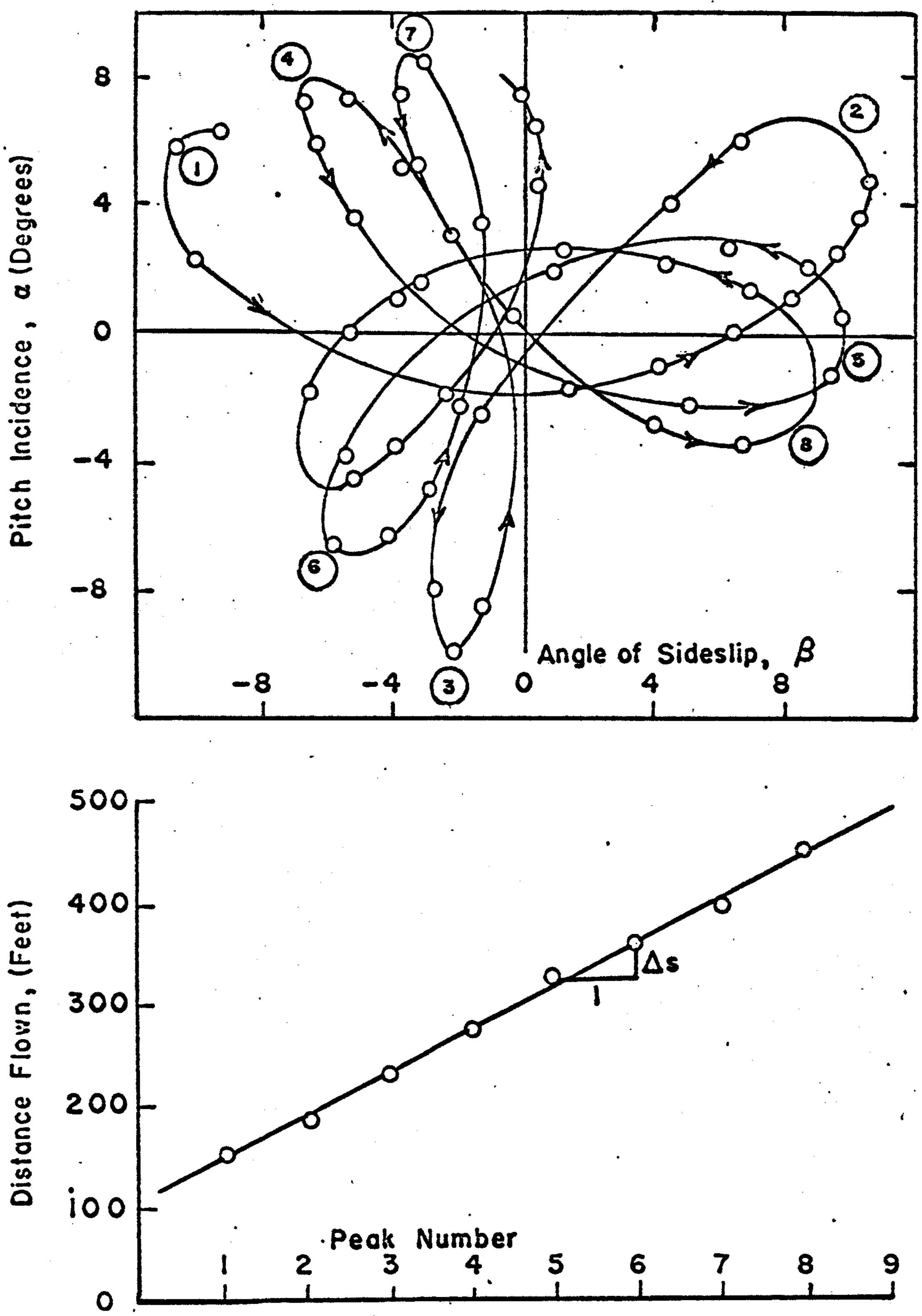
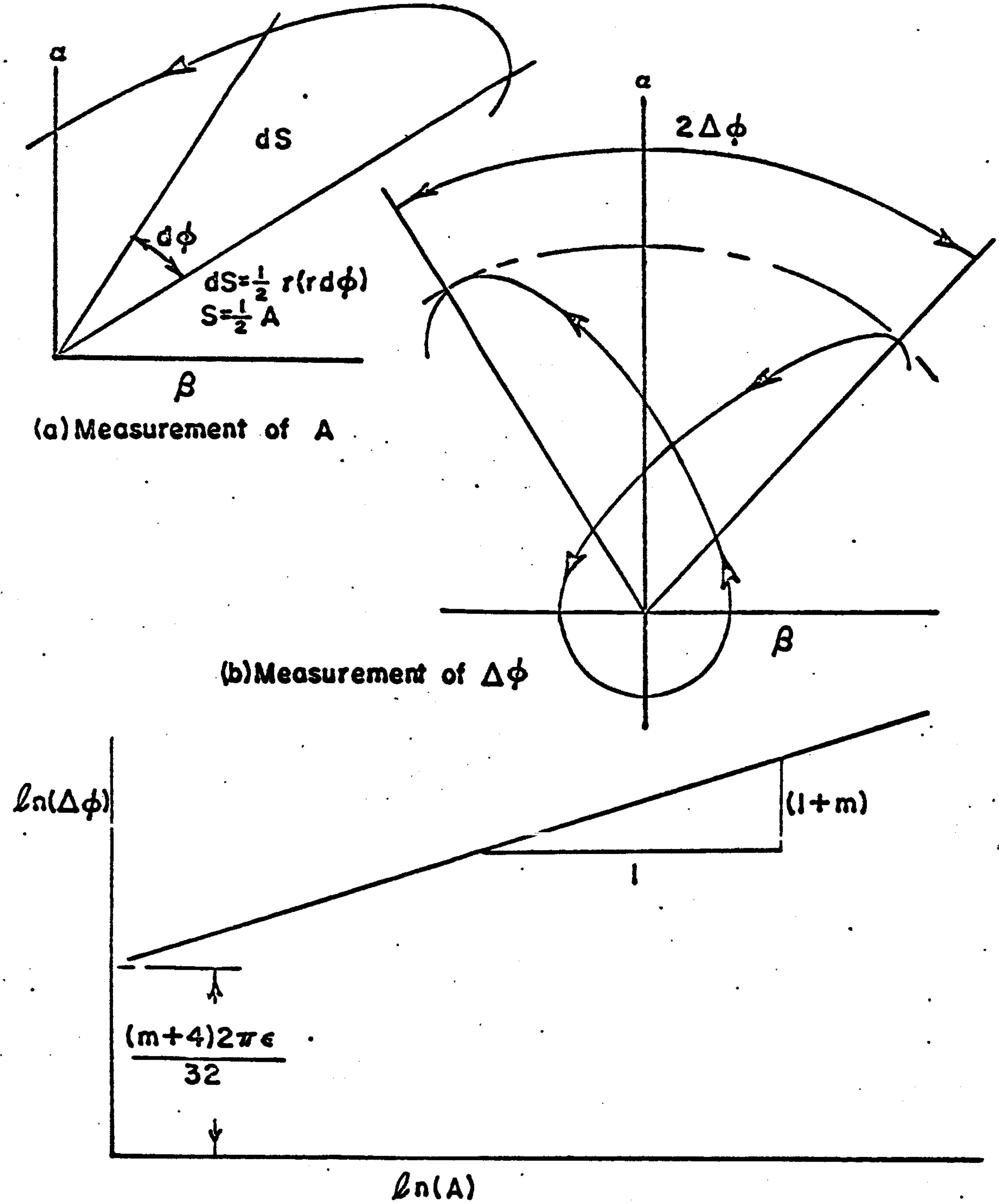


FIG. 6 EXTRACTION OF STATIC STABILITY PARAMETER(C_{m_α}) FROM LINEAR DATA



(a) Measurement of A

(b) Measurement of Δφ

(c) Measurement of ε, m

FIG. 7 EXTRACTION OF STABILITY PARAMETERS FROM NON-LINEAR DATA